SEPT 2024 TOPOLOGY PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

Date: September 17, 2024.

(1) Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. For $c \in \mathbb{R}$, let $Y_c = \{(x, y) \in \mathbb{R}^2 \mid x - y = c\}$.

(a) Give formulas for the tangent spaces $T_{(x,y)}X$ for $(x, y) \in X$ and $T_{(x,y)}Y_c$ for $(x, y) \in Y_c$.

(b) Determine and prove for which values of $c \in \mathbb{R}$ does X intersect Y_c transversally in $\mathbb{R}^2$?

(2) Suppose X and Y are smooth, connected, oriented manifolds (no boundary), and $\dim(X) = \dim(Y)$. Let $f : X \to Y$ be a smooth map.

(a) Assume X is compact. Suppose there is a point $y_0 \in Y$ such that $f^{-1}(y_0) = \{x_0\}$ and $df_{x_0} : T_{x_0}X \to T_{y_0}Y$ is an isomorphism of vector spaces. Prove that f is surjective.

(b) Give an example of a map $f : \mathbb{R} \to \mathbb{R}$, such that there exists a point $y \in \mathbb{R}$ with a single point x in its preimage ($f^{-1}(y) = \{x\}$), but such that f is not surjective. (Verify it satisfies the claimed properties.)

(3) Consider the torus $T^2 \subset \mathbb{R}^3$ given by

$$T^2 = \{((\cos\theta + 2)\cos\phi, (\cos\theta + 2)\sin\phi, \sin\theta) \mid (\theta, \phi) \in [0, 2\pi) \times [0, 2\pi)\}.$$

Let $i : T^2 \to \mathbb{R}^3$ denote the inclusion map. Let ω be the 2-form on $\mathbb{R}^3$ given by

$$\omega = x\,dx \wedge dy + (y + y^2)\,dy \wedge dz + (z + z^2)\,dx \wedge dz$$

Prove that

$$\int_{T^2} i^*\omega = 0.$$

(4) Covering space examples:

(a) Let W be the wedge of two circles. Draw **two** connected 2-fold covering spaces of W that are not homeomorphic.

(b) Let F be a closed orientable surface of genus 5. Show that any two connected 2-fold covering spaces of F are homeomorphic.

(5) Let X be the topological space obtained from $\mathbb{R}^3$ by removing the x-axis and the y-axis. Compute the fundamental group of X.

(6) Let $X = \mathbb{R}P^2 \times \mathbb{R}P^2$.

(a) Compute $\pi_1(\mathbb{R}P^2)$ and $\pi_1(X)$. (Justify your computations.)

(b) What is the universal cover $\tilde{X}$ of X? How does $\pi_1(X)$ acts as a group of covering transformations on $\tilde{X}$? (Specify what are the covering (deck) transformations that each generator of $\pi_1(X)$ is sent to.)