ANALYSIS PRELIM EXAM

(1) Let f be a continuous function on $[0,1]$. Assume $\int_0^1 x^n f(x)dx = 0$ for all $n \in \mathbb{N}_0 = \{0,1,2,3,\dots\}$. Prove f is identically zero on $[0,1]$.

(2) Let $(X, ||\cdot||)$ be a Banach space. Suppose $x_n \in X$ is a sequence such that $||x_n|| \le 1$.

  (a) If $x_n \to x$ in the norm topology, prove that $||x|| \le 1$.

  (b) If $x_n \rightharpoonup x$ weakly in X, prove that $||x|| \le 1$.

(3) Let $X = C([0,1])$ endowed with the uniform norm, $||\cdot||_\infty$, and let $Y = C([0,1])$ endowed with the norm $||f||_1 := \int_0^1 |f(x)|dx$. Define the operators $Tf(x) = \frac{1}{x}\int_0^x f(t)dt$ and $If(x) = \int_0^x f(t)dt$. Prove

  (a) $I : X \to X$ is compact.

  (b) $T : Y \to X$ is **not** bounded.

(4) Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space.

  (a) Write a definition of an orthonormal basis.

  (b) Show that any orthonormal basis is countable.

  (c) Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis. Show that for every $x \in \mathcal{H}$

$$||x||^2 = \sum_{n=1}^{\infty} |\langle e_n, x\rangle|^2.$$

  The last statement is called the Parseval's equality. (For full credit we expect a proof starting from the definition you wrote in part 4a.)

(5) An operator L on a Hilbert space is called nilpotent if there exists $n \in \mathbb{N}$ such that $L^n = 0$ and quasi-nilpotent if the spectrum of L contains only zero, $\sigma(L) = \{0\}$.

  (a) Show that every nilpotent operator is quasi-nilpotent,

  (b) Show that if L is Hermitian and quasi-nilpotent then $L = 0$,

  (c) Let $L : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $(Lx)_k = 2^{-(k+1)} x_{k+1}$. Prove that L is quasi-nilpotent, but not nilpotent.

(6) Let

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{-inx} dx,$$

and define

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

  (a) Find the values $f(1)$ and $f(0)$.

  (b) Does $f \in L^2[(-\pi,\pi)]$? If yes, compute the norm.

  (c) Does $f \in C([-\pi,\pi])$? If yes, compute the norm.

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