SEPT 2024 ANALYSIS PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

Date: September 16, 2024.

(1) Let X, Y, Z be Banach spaces. Let $B(X, Y)$ be the Banach space of bounded linear operators mapping $X \to Y$ equipped with the standard operator norm. Likewise, $B(Y, Z)$ is the space of bounded linear operators $Y \to Z$. Let $T \in B(X, Y)$ and $S \in B(Y, Z)$.

(a) Prove that T is continuous (this means that if $x_n \to x$ in X, then $Tx_n \to Tx$ in Y).

(b) Prove that $\|ST\| \leq \|S\|\|T\|$.

Note: You must start with the definition of a **bounded** linear operator to receive full credit, you cannot simply state that both (a) and (b) are known lemmas.

(2) Consider the space, $\ell_0^\infty(\mathbb{N})$, of sequences $x : \mathbb{N} \to \mathbb{R}$ that have zero limit, *i.e.* $\ell_0^\infty(\mathbb{N}) = \{x : \lim_{n\to\infty} x_n = 0\}$. Equip this space with the norm $\|x\| = \sup_n |x_n|$. Define operators $S, T : \ell_0^\infty(\mathbb{N}) \to \ell_0^\infty(\mathbb{N})$ by

$$(Tx)_n := \frac{x_n}{n}, \quad (Sx)_n := \frac{n-1}{n} x_n.$$

(a) Compute the operator norm $\|T\|$.

(b) Prove that T is a compact operator.

(c) Prove that S is NOT a compact operator.

(3) Define the vector space $X = \{f : [0, 1] \mapsto \mathbb{R} : f \text{ continuous}, f(0) = 0\}$. Let $0 < \alpha < 1$. Define the norms

$$\|f\| := \sup_{0\leq x\leq 1} |f(x)|, \qquad \|f\|_\alpha := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Consider the Banach spaces $C_0([0, 1]) = (X, \|\cdot\|)$ and $C_0^\alpha([0, 1])] = (X, \|\cdot\|_\alpha)$.

(a) For any $0 < \alpha < 1$, prove that $\mathcal{B}_\alpha := \{f \in X : \|f\|_\alpha \leq 1\}$ is precompact in $C_0([0, 1])$.

(b) Let $0 < \alpha < \beta < 1$. Prove that there exists a constant $C < \infty$ such that for all $f \in X$, the interpolation inequality,

$$\|f\|_\alpha \leq C\|f\|_\beta^{\frac{\alpha}{\beta}} \|f\|^{1-\frac{\alpha}{\beta}},$$

holds.

(c) Let $0 < \alpha < \beta < 1$. Prove that $\mathcal{B}_\beta$ is precompact in $C_0^\alpha([0, 1])$. **Hint:** use parts (a) and (b).

(4) Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\{e_n\}_{n=1}^\infty$ an orthonormal basis. Show that the map $S : \mathcal{H} \to l^2(\mathbb{N})$,

$$x \mapsto (Sx)_n = \langle e_n, x\rangle,$$

is an isometry. This means that you need to show that S is a bijection and that $\|Sx\| = \|x\|$ holds for all $x \in \mathcal{H}$.

(5) Let T be a compact Hermitian operator on a Hilbert space. Assume that

$$e^{2\pi iT} = \mathrm{Id}.$$

(a) Show that $\sigma(T) \subset \mathbb{Z}$,

(b) Show that T is a finite rank operator.

Note: For any $z \in \mathbb{C}$, e^{zT} is defined via functional calculus or by the series

$$e^{zT} = \sum_{n=0}^{\infty} \frac{(zT)^n}{n!}.$$

(6) Write the Fourier series of the function $f : [0, 2\pi) \to \mathbb{R}$,

$$f(x) = x.$$

Does the series converge pointwise? If yes, what is the limit? Does the series converge in the topology of $L^2([0, 2\pi))$?