ALGEBRA PRELIM EXAM

(1) What are the possible isomorphism types of the subgroups of the abelian group $(\mathbb{Z}/5) \oplus (\mathbb{Z}/25)$?

(2) Does the free group F_2 on two generators have a non-trivial finite subgroup?

(3) Give an example of a commutative ring R which is not Noetherian. Prove that your example satisfies the claimed properties.

(4) Let R be a ring, and let

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\phi} M \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P' \xrightarrow{\phi'} M \longrightarrow 0$$

be short exact sequences of left R-modules, where P and P' are projective R-modules.

Prove that $P \oplus K' \cong P' \oplus K$ as R-modules.

Hint: Let X denote the fiber product of ϕ and ϕ'. Show that there is a short exact sequence

$$0 \longrightarrow K' \longrightarrow X \longrightarrow P \longrightarrow 0\ .$$

(5) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ as left $\mathbb{Q}$-modules.

(6) Let p be a prime.

(a) Show that the Galois group of $x^5 - 1 \in \mathbb{F}_p[x]$ depends only on $p \mod 5$.

Hint: Determine the factorization of $f(x) = x^5 - 1$ into irreducibles over $\mathbb{F}_p$ in each case.

(b) Compute the Galois group for each congruence class $p \mod 5$.

If you have completed part (a), part (b) should be short. If you haven't completed part (a), you may still compute part (b).

Date: March 27, 2024.