SEPT 2024 ALGEBRA PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

Date: September 19, 2024.

(1) Let G be a finite group with $22,893,266 = 2 \cdot 11 \cdot 101 \cdot 10303$ elements, with the indicated prime factorization. Prove that G is a solvable group.

(2) Prove that if F is a non-abelian free group, then its center $Z(F)$ is trivial.

(3) Is the quotient map $f : \mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$ a monomorphism between commutative rings? In other words, if $g, h : R \to \mathbb{Z}$ are two ring homomorphisms from another commutative ring R and $f \circ g = f \circ h$, does that imply that $g = h$?

(4) Let R be a Noetherian commutative domain with the property that every $n \times k$ matrix M can be put in Smith normal form. (The means that $M = XDY$ where X and Y are invertible matrices defined over R, and the matrix entries of D satisfy $D_{a,b} = 0$ when $a \neq b$ and $D_{a,a} \mid D_{a+1,a+1}$.) Prove that R is a principal ideal domain.

(5) Find an example of a finite field extension K/F and an **algebraic** $\zeta \in \overline{K}$ such that $[K(\zeta) : F(\zeta)]$ does **not** divide $[K : F]$.

(6) Determine the Galois group of $f(x) = x^5 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$, where $\mathbb{F}_2$ is the field with 2 elements.