ALGEBRA PRELIM EXAM

(1) A group has sixty elements including at least two of order five which do not commute. How many of the elements have order five?

(2) Take $\alpha = 2^{\frac{1}{3}} + 3^{\frac{1}{2}} \in \mathbb{R}$.

  (a) Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(2^{\frac{1}{3}}, 3^{\frac{1}{2}})$

  (b) Find $\dim_K H$ if K is the subfield of $\mathbb{R}$ generated by α and $H \supseteq K$ is a splitting field for α.

(3) For how many values of $a \in \mathbb{F}_q$ is the ring $\mathbb{F}_q[x]/(x^3 - a)$ another field if:

  (a) $q = 11$

  (b) $q = 13$

(4) Let $S = \mathbb{R}\langle x, y\rangle$ (this is the noncommutative polynomial ring, also equal to the tensor algebra $T(\mathbb{R}^2)$).

Let $M = \mathbb{R}[x]$ be the S-module defined via

- x acts as left multiplication by x
- y acts as $\frac{d}{dx}$.

Prove M is a simple S-module.

(5) Let R and S be commutative rings, let $\varphi : R \to S$ be a ring homomorphism.

For the following statements, if true prove it; if false give a counter-example.

  (a) If $Q \subseteq S$ is a prime ideal, then the ideal $\varphi^{-1}(Q)$ is a prime ideal of R.

  (b) If $J \subseteq S$ is an ideal and $\varphi^{-1}(J)$ is a prime ideal of R, then J is a prime ideal of S.

  (c) If φ is surjective, if $I \subseteq R$ is an ideal, and if $\varphi(I)$ is a prime ideal of S, then I is a prime ideal of R.

(6) Give an example of a ring R and a left R-module M such that M is a projective R-module but not a free R-module.

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