SEPTEMBER 2025 ANALYSIS PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support and L^p-norm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f|^p \, dx \right)^{1/p}.$$

If $1 < p < r < q < \infty$, prove that

$$\|f\|_r \leq \|f\|_p^{\theta} \|f\|_q^{1-\theta},$$

where $0 < \theta < 1$ satisfies

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

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(2) Let $\mathcal{H}$ be a Hilbert space. A sequence (A_n) of bounded linear operators $A_n : \mathcal{H} \to \mathcal{H}$ is said to *converge strongly* to $A : \mathcal{H} \to \mathcal{H}$ (written $A_n \to A$) if $A_n x \to Ax$ strongly in $\mathcal{H}$ (i.e., with respect to the $\mathcal{H}$-norm) for every $x \in \mathcal{H}$, and *converge weakly* (written $A_n \rightharpoonup A$) if $A_n x \rightharpoonup Ax$ weakly in $\mathcal{H}$ for every $x \in \mathcal{H}$.

(a) Prove that $A_n \to A$ implies that $A_n^* \rightharpoonup A^*$, where A^* denotes the Hilbert-space adjoint of A.

(b) Let $S_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the n-fold left-shift operator on the sequence space $\ell^2(\mathbb{N})$, defined by $S_n(x_1, x_2, x_3, \dots) = (x_{n+1}, x_{n+2}, x_{n+3}, \dots)$. Prove that (S_n) converges strongly but (S_n^*) does not converge strongly.

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(3) Prove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there is a sequence (p_n) of polynomials such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every $k = 0, 1, 2, \ldots$. Here, $f^{(k)}$ denotes the kth derivative of f.

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(4) Consider the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ acting on a function f that belongs to the Schwarz space as

$$\mathcal{F}(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx.$$

(a) Show that $(\mathcal{F}^2 f)(x) = f(-x)$.

(b) Find the spectrum of $\mathcal{F}$, $\sigma(\mathcal{F})$.

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(5) Write the Fourier series of the function $f : [-\pi, \pi) \to \mathbb{R}$,

$$f(x) = \pi - |x|.$$

(a) Does the series converge point-wise? if yes, what is the limit?

(b) Does the series converge uniformly?

(c) Does the series converge in the topology of $L^2([-\pi, \pi))$?

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(6) Let H be a bounded self-adjoint operator on a Hilbert space. Prove that $(H - i)$ is invertible and show that

$$U = (H - i)^{-1}(H + i)$$

is an isometry.

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