# SEPTEMBER 2025 ALGEBRA PRELIM EXAM

**Name:**

**Student ID:**

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

(1) Prove that the semidirect product of two solvable groups is solvable.

.

(2) Determine, with justification, the composition factors of the permutation group $S_4$.

.

(3) Let $E$ be a Galois extension of $F$ with Galois group $G$, and let $L$ be the fixed field of a subgroup $H$ of $G$. Show that the automorphism group of $L/F$ is $N/H$ where $N$ is the normalizer of $H$ in $G$.

.

(4) Consider the tensor product

$$V = \mathbb{Q}[x]/(x^4 - 4x^2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x^3 + x^2 - 4x - 4).$$

(a) In particular, $V$ is a $\mathbb{Q}$-module. Compute $\dim_{\mathbb{Q}} V$ .

(b) $V$ is also a $\mathbb{Q}[x]$-module. Let $A$ be the linear transformation that is multiplication by $x$ on $V$. Give the matrix that is the rational canonical form of $A$.

.

(5) Let $\mathbb{F}$ be a field.
For the computations below, note your answer is best given as an $\mathbb{F}[x]$-module, but if unable to give an answer in that form, substantial partial credit will be given to describing the answer as an $\mathbb{F}$-module, (that is, as an $\mathbb{F}$-vector space).

(a) Compute $\mathrm{Hom}_{\mathbb{F}[x]}(\,\mathbb{F}(x), \mathbb{F}[x]\,)$

(b) Compute $\mathrm{Hom}_{\mathbb{F}[x]}(\,\mathbb{F}[[x]], \mathbb{F}[x]\,)$

.

(6) Does there exist a ring homomorphism $g : \mathbb{Z}[i] \to Z/10\mathbb{Z}$ ? If not, explain why not. If yes, pick one such $g$ and compute the ideal $\ker(g)$.

.

.