SEPTEMBER 2025 TOPOLOGY PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, append the extra page to the end of your exam, write a note saying continued on extra page in the page within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page.

(1) Consider the subset

$$X_\alpha := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - \alpha)^2 + z^2 = 1\}$$

(a) Determine the values of $\alpha \in \mathbb{R}$ such that X_α is a smooth manifold.

(b) Compute the tangent space of X_{10} at any points $(x, y, z) \in X$ with $z = 0$ and provide two basis vectors of this tangent space at each such point.

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(2) Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \in \Omega^2(\mathbb{R}^4)$.

(a) Consider the surface $L := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = 3, \quad x_3^2 + x_4^2 = 5\}$ and compute $\int_L \omega$.

(b) Determine for which parameters $\lambda \in \mathbb{R}$ the restriction of ω to the 2-plane

$$\Pi_\lambda := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 = \lambda x_2, x_4 = 0\}$$

is a non-zero 2-form at every point of Π_λ.

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(3) Consider $X = \mathbb{R}^3 \setminus \{0\}$ with Cartesian coordinates (x, y, z), and the 2-form

$$\omega = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \cdot (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy) \in \Omega^2(X).$$

(a) Show that ω is closed.

(b) Prove that ω is not exact.

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(4) Give an explicit homotopy equivalence between X and Y given as follows. Include a detailed computation proving that your map is indeed a homotopy equivalence.

$$X = [0,1]/\sim$$

where $\sim$ is defined by $t \sim t'$ if and only if $t = t'$ or $\{t, t'\} = \{0, 1\}$. Here, $[0,1]$ has the subspace topology of $\mathbb{R}$ with the Euclidean topology, and $[0,1]/\sim$ has the quotient topology.

$$Y = \mathbb{R}^2 \setminus \{(0,0)\}$$

where Y has the subspace topology of $\mathbb{R}^2$ and $\mathbb{R}^2$ has the Euclidean topology.

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(5) Fundamental group and covering spaces of $\mathbb{R}\mathrm{P}^2 \vee S^1$

(a) Show that the fundamental group of $\mathbb{R}\mathrm{P}^2 \vee S^1$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$, the free product of the group of order two with the infinite cyclic group.

(b) For each of the following subgroups of $\mathbb{Z}/2 * \mathbb{Z}$, give a pointed covering map $p : \tilde{X} \to \mathbb{R}\mathrm{P}^2 \vee S^1$ such that the covering group $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is the given subgroup and $\tilde{X}$ is connected.

(i) $\mathbb{Z}/2\mathbb{Z} * 1 \subset \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$

(ii) the cyclic subgroup of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$ generated by ab where $a \in \mathbb{Z}/2\mathbb{Z}$ is the generator of $\mathbb{Z}/2\mathbb{Z}$ and $b \in Z$ is the generator of $\mathbb{Z}$

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(6) Let E be a (locally trivial) fiber bundle over an orientable genus 2 surface Σ with fiber S^1. Prove that $\pi_n(E, e_0) = 0$ for all $n \geq 2$.

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