JUNE 2026 ALGEBRA PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

(1) Show that if H is a proper subgroup of G and $|G|$ does not divide $[G : H]!$, then G cannot be simple. *Note that "!" denotes factorial here.*

(2) Prove that a nontrivial finite group is solvable if and only if all of its composition factors have prime order.

(3) a) Find the splitting field K of $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.

b) Find the Galois group of K over $\mathbb{Q}$.

(4) Find an ideal $I \subset \mathbb{Q}[x]$ such that
$$\mathbb{Q}[x]/(x^2 - x - 2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x^2 - 4)$$
is isomorphic to $\mathbb{Q}[x]/I$. Describe the isomorphism using the universal property of tensor products.

(5) Suppose we have an exact sequence of $R-$modules

$$0 \to \mathbb{Z}/4 \to M \to \mathbb{Z}/3 \to 0,$$

where $R = \mathbb{Z}/12$, and the action is by multiplication. Prove that M is isomorphic to $R = \mathbb{Z}/12$ as an R-module.

(6) Describe all prime ideals in $\mathbb{R}[x]/(x^3 - 1)$. Which ones are maximal?