JUNE 2026 ANALYSIS PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

(1) Let $\ell^2(\mathbb{N})$ consist of sequences $a = (a^{(1)}, a^{(2)}, \dots)$ such that $\sum_{n=1}^{\infty} |a^{(n)}|^2 < \infty$, and define the norm $\|a\|_{\ell^2} := \sqrt{\sum_{n=1}^{\infty} |a^{(n)}|^2}$.

(a) State the definition of a sequence $\{x_n\}_{n=1}^{\infty}$, where each $x_n \in \ell^2$, converging strongly to 0.

(b) State the definition of a sequence $\{x_n\}_{n=1}^{\infty}$, where each $x_n \in \ell^2$, converging weakly to 0.

(c) Prove that if a sequence $\{x_n\}_{n=1}^{\infty}$, where each $x_n \in \ell^2$, converges strongly to 0 then it converges weakly to 0.

(d) For each $n \in \mathbb{N}$, let e_n be the standard basis vector in ℓ^2 (that is e_n has a 1 in the $n'th$ slot and 0 elsewhere). Prove that $\{e_n\}_{n=1}^{\infty}$ converges weakly to zero.

(e) Prove that $\{e_n\}_{n=1}^{\infty}$ does not converge strongly to zero.

(2) Consider $C([0,1])$ with the uniform norm, $\|\cdot\|_\infty$.

(a) Let $\mathcal{F} \subset C([0,1])$. Write down the following definitions: $\mathcal{F}$ is **bounded**, $\mathcal{F}$ is **uniformly equicontinuous**. Also state the **sequential characterization of compactness** of $\mathcal{F}$.

(b) State the Arzela-Ascoli Theorem.

(c) Consider the following subsets of $C([0,1])$:

$$\mathcal{F}_1 := \{\sin(2\pi n x); n \in \mathbb{N}\},$$
$$\mathcal{F}_2 := \{x + n; n \in \mathbb{N}\},$$
$$\mathcal{F}_3 := \{\frac{1}{n}\sin(2\pi n^2 x); n \in \mathbb{N}\}.$$

For each $j = 1, 2, 3$, decide if $\mathcal{F}_j$ is bounded, if $\mathcal{F}_j$ is uniformly equicontinuous, and if $\mathcal{F}_j$ is compact. Give a **complete proof** of your claims.

(3) Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Consider the operator

$$Tf(x) = \int_0^1 k(x,y)f(y)\,dy.$$

Let $X = C([0,1])$ with the uniform norm $\|\cdot\|_\infty$, and let $Y = C([0,1])$ with the norm $\|f\|_1 := \int_0^1 |f(x)|dx$. Prove that

(a) T is a bounded operator from Y to X.

(b) T is a compact operator from X to X.

(c) T is a compact operator from X to Y.

(4) Let

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ \sin x, & 0 \le x \le \pi, \end{cases}$$

and extend f to be 2π-periodic.

(a) Find the Fourier series of f.

(b) Determine where $f(x)$ is equal to its Fourier series. Remember to justify your work.

(5) Let

$$X = C([0,1];\mathbb{C})$$

be the complex Banach space of continuous complex-valued functions on $[0,1]$, equipped with the sup norm

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|.$$

Define $T : X \to X$ by

$$(Tf)(x) = \int_0^x y f(y)\, dy + x \int_x^1 f(y)\, dy.$$

(a) Prove that T is a linear, bounded, and compact operator on X.

(b) Prove that

$$\sigma(T) = \{0\} \cup \left\{ \frac{4}{(2m+1)^2\pi^2} : m = 0, 1, 2, \dots \right\}.$$

Determine the eigenvalues of T.

(6) Let $u \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Define the Fourier transform by

$$\widehat{u}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} u(x)\, dx.$$

(a) Show that

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|\widehat{u}\|_{L^1(\mathbb{R})}.$$

(b) Use the Cauchy-Schwarz inequality to prove that, for every $R > 0$,

$$\|\widehat{u}\|_{L^1(\mathbb{R})} \leq (2R)^{1/2}\|\widehat{u}\|_{L^2(\mathbb{R})} + \left(\frac{2}{R}\right)^{1/2} \|\xi\widehat{u}\|_{L^2(\mathbb{R})}.$$

Then choose $R > 0$ to show that

$$\|\widehat{u}\|_{L^1(\mathbb{R})} \leq 10\|u\|_{L^2(\mathbb{R})}^{1/2}\|u'\|_{L^2(\mathbb{R})}^{1/2}.$$

Conclude that

$$\|u\|_{L^\infty(\mathbb{R})} \leq 10\|u\|_{L^2(\mathbb{R})}^{1/2}\|u'\|_{L^2(\mathbb{R})}^{1/2}.$$