

SEPTEMBER 2026 ALGEBRA PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

- (1) Let M, N be normal solvable subgroups of a group G . Show that their product MN is also solvable.

- (2) Suppose G is a simple group with $|G| = 168 = 23 \cdot 3 \cdot 7$. How many elements of order 7 does G contain? Justify your answer.

- (3) a) A finite Galois extension K of a field F has degree 160. Show that there is an intermediate field L between K and F such that $[L : F] = 5$.
- b) Let K be a finite normal extension of F and let E be the fixed field of the group of all F -automorphisms of K . Show that the minimal polynomial over F of each element of E has only one distinct root.

(4) Let $R = \mathbb{Q}[t]/(t^4 - 1)$, $I = (t^2 + 1)$ and $J = (t - 1)$, and suppose that

$$0 \longrightarrow R/I \longrightarrow M \longrightarrow R/J \longrightarrow 0$$

is an exact sequence of R -modules. Is M necessarily isomorphic to $R/I \oplus R/J$? Prove your answer.

- (5) Let $R = \text{Mat}_n(\mathbb{Q})$ be the $\mathbb{Q}$ -algebra of all $n \times n$ matrices. Let $M \cong \mathbb{Q}^n$ be the vector space of row vectors, regarded as a right R -module via right multiplication, and let $N \cong \mathbb{Q}^n$ be the vector space of column vectors, regarded as a left R -module via left multiplication. Prove that

$$M \otimes_R N \cong \mathbb{Q}$$

as abelian groups, using the universal property of tensor products.

- (6) Answer the following questions.
- (a) State what it means for an element of an integral domain to be irreducible, and what it means for it to be prime.
 - (b) Prove that in a principal ideal domain every irreducible element is prime.
 - (c) Is $R = \mathbb{Q}[t^2, t^3]$ a principal ideal domain? Prove your answer.

