

SEPTEMBER 2026 ANALYSIS PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

- (1) Let $\ell^2(\mathbb{N})$ denote the Hilbert space of square-summable sequences, $x = (x^{(1)}, x^{(2)}, \dots)$, endowed with the norm $\|x\|_{\ell^2} := (\sum_{k=1}^{\infty} |x^{(k)}|^2)^{\frac{1}{2}}$. Let $f_n = (1, 1, \dots, 1, 0, 0, 0, \dots)$ denote the element of ℓ^2 whose first n terms are 1 and whose remaining terms are zero. For each $n \in \mathbb{N}$, define $a_n \in \ell^2$ via

$$a_n = \frac{1}{n^\alpha} f_n, \quad 0 < \alpha < 1.$$

Classify the $\alpha \in (0, 1)$ for which the sequence $(a_n)_{n=1}^{\infty} \subset \ell^2$ converges strongly, weakly but not strongly, and neither weakly nor strongly. Give **complete proofs** of your claims.

- (2) Consider $C([0, 1])$ with the uniform norm, $\|\cdot\|_\infty$.
- (a) Let $\mathcal{F} \subset C([0, 1])$. Write down the following definitions: $\mathcal{F}$ is **bounded**, $\mathcal{F}$ is **uniformly equicontinuous**. Also state the **sequential characterization of compactness** of $\mathcal{F}$.
- (b) State the Arzela-Ascoli Theorem.
- (c) Define the Lipschitz seminorm

$$[f]_{\text{Lip}} := \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|.$$

Consider the following subsets:

$$\mathcal{F}_1 := \{f \in C([0, 1]) : [f]_{\text{Lip}} \leq 1\},$$

$$\mathcal{F}_2 := \{f \in C([0, 1]) : [f]_{\text{Lip}} \leq 1 \text{ and } f(0) = 0\},$$

$$\mathcal{F}_3 := \{f \in C([0, 1]) : [f]_{\text{Lip}} < 1 \text{ and } f(0) = 0\},$$

$$\mathcal{F}_4 := \{f_n(x) = x^n : n \in \mathbb{N}\}.$$

Which of these are compact subsets of $C([0, 1])$? Give a **complete proof** of your claims.

- (3) Let $\ell^2(\mathbb{N})$ denote the Hilbert space of square-summable sequences, $x = (x^{(1)}, x^{(2)}, \dots)$, endowed with the norm $\|x\|_{\ell^2} := (\sum_{k=1}^{\infty} |x^{(k)}|^2)^{\frac{1}{2}}$. Define two operators $T, S : \ell^2 \rightarrow \ell^2$ via

$$(Tx)^{(n)} = \frac{x^{(n)}}{n}, \quad S(x^{(1)}, x^{(2)}, x^{(3)}, \dots) = (0, x^{(1)}, x^{(2)}, x^{(3)}, \dots).$$

Prove that

- (a) T is bounded and compute its operator norm.
- (b) T is compact.
- (c) S is **not** compact.

(4) Let

$$f(x) = |\sin x|,$$

viewed as a 2π -periodic function.

(A) Find the Fourier series of f .

(B) Determine where $f(x)$ is equal to its Fourier series.

(5) Let

$$X = C([0, 1]; \mathbb{C})$$

be the Banach space of continuous complex-valued functions on $[0, 1]$, equipped with the sup norm

$$\|f\|_{\infty} = \max_{x \in [0, 1]} |f(x)|.$$

Define $T : X \rightarrow X$ by

$$(Tf)(x) = \int_0^{1-x} f(y) dy, \quad x \in [0, 1].$$

(a) Prove that T is a linear, bounded, and compact operator on X . Determine $\|T\|$.

(b) Prove that

$$\sigma(T) = \{0\} \cup \left\{ \frac{2}{(4\ell + 1)\pi} : \ell \in \mathbb{Z} \right\}.$$

where $\sigma(T)$ represents the spectrum of T .

- (6) Let $u \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Define the Fourier transform by

$$\widehat{u}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} u(x) dx.$$

- (A) Use integration by parts to prove that

$$\widehat{u}'(\xi) = 2\pi i \xi \widehat{u}(\xi).$$

Conclude that

$$\|\xi \widehat{u}\|_{L^2(\mathbb{R})} = \frac{1}{2\pi} \|u'\|_{L^2(\mathbb{R})}.$$

- (B) Show that for every $R > 0$,

$$\int_{|\xi| \leq R} |\widehat{u}(\xi)|^2 d\xi \leq 2R \|u\|_{L^1(\mathbb{R})}^2.$$

- (C) Show that for every $R > 0$,

$$\int_{|\xi| > R} |\widehat{u}(\xi)|^2 d\xi \leq \frac{1}{4\pi^2 R^2} \|u'\|_{L^2(\mathbb{R})}^2.$$

- (D) Combine Parts (B) and (C), then choose $R > 0$ to prove

$$\|u\|_{L^2(\mathbb{R})}^6 \leq 10 \|u\|_{L^1(\mathbb{R})}^4 \|u'\|_{L^2(\mathbb{R})}^2.$$

