

SEPTEMBER 2026 TOPOLOGY PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

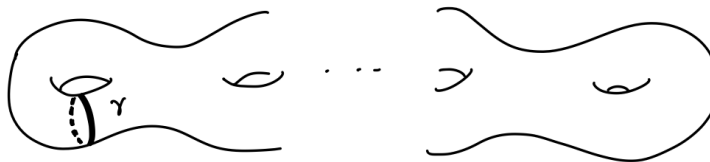
Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, append the extra page to the end of your exam, write a note saying continued on extra page in the page within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page.

- (1) Let $V = S^1 \times D^2$ be a solid torus, and let M be a Möbius band, shown below.



Prove that V and M are homotopy equivalent.

- (2) Let $F = S^1 \times S^1 - D^2$ be a punctured torus, and let Σ_g be a closed, connected, orientable surface of genus g , shown below.



Let γ a *meridian* of Σ , draw as the thick loop on Σ_g above. Let X be the surface obtained by gluing F to Σ_g by identifying ∂F and γ . Compute $\pi_1(X)$.

- (3) Construct a nonnormal covering of the Klein bottle by the torus (and justify your assertions).

- (4) Consider the subset $X_c = \{x^2 + y^2 - z^2 = c\}$ in $\mathbb{R}^3$.
- a) Find all values of c such that X_c is a smooth manifold. Find the dimension of X_c in these cases.
 - b) Find all values of c such that X_c intersects the line $\{x = y = z\}$ transversally. For which c the intersection is not empty?

(5) Suppose $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function. Consider the map

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad F(x, \theta) = (f(x) \cos(\theta), f(x) \sin(\theta), x)$$

- a) Compute the rank of the map $dF : T_{(x, \theta)}\mathbb{R}^2 \rightarrow T_{F(x, \theta)}\mathbb{R}^3$ at all points.
- b) Find all points (x, θ) where dF is injective.
- c) Find all points (x, θ) where dF is surjective.

(6) Consider the upper half-sphere $S^+ = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$, and the 2-form $\alpha = z^5 dx \wedge dy$.

a) Consider the map

$$P : [0, 1] \times [0, 2\pi] \rightarrow S^+, \quad P(r, \varphi) = (r \cos(\varphi), r \sin(\varphi), \sqrt{1 - r^2}).$$

Compute the pullback $P^*\alpha$.

b) Compute the integral $\int_{S^+} \alpha$.

