GGAM Prelim Questions - Fall 2013

Instructions:

• All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

• Use separate sheets for the solution of each problem.

1. Consider the second order ODE which describes the height, \( h(x) \), of a wave in a medium with varying wave speed

\[
\frac{d^2 h}{dx^2} + k^2(x) h = 0
\]

where \( k(x) \) is the local wavenumber

\[
k(x) = k_1 + (k_2 - k_1) \tanh(x/L), \quad 0 < k_1 < k_2.
\]

(a) Non-dimensionalize the system by using \( L \) to measure length.

(b) For \( L \gg 1 \), write down the first two leading order (i.e. eikonal and transport) equations for the WKB approximation of \( h(x) \) (Do this by considering a general form of \( k(x) \) - you need not substitute the particular \( k(x) \)).

(c) Suppose the wave profile is asymptotically

\[
h(x) = Ae^{ik_1 x} \quad \text{for} \quad x \to -\infty.
\]

Solve the WKB equation(s) to determine the asymptotic profile \( |h(x)| \) for \( x \to \infty \).

2. The function \( y(x; \epsilon) \) satisfies

\[
\epsilon y'' + \sqrt{x} y' + y = 0 \quad \text{in} \quad 0 \leq x \leq 1
\]

with boundary conditions \( y(0) = 0 \), and \( y(1) = 1 \). Find the matched asymptotic (inner and outer) solutions.

3. The small, centrally symmetric vibrations of a stretched uniform circular membrane, fixed round its perimeter, are approximately described by the equations

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = a^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), & 0 \leq r \leq R, \ t \geq 0; \\
u(R, t) = 0, & t \geq 0.
\end{cases}
\]

Here \( R \) is the radius of the membrane, \( u(r, t) \) is the transverse displacement of a point distant \( r \) from the center of the membrane at time \( t \), and \( a \) is a positive constant.

(a) Separate variables to obtain a singular Sturm-Liouville system.

(b) Find the eigenvalues of this system in terms of the zeros of the Bessel function \( J_0 \), and write down the corresponding eigenfunctions.
4. Consider the following regular Sturm-Liouville (RSL) problem:

\[
\begin{cases}
  f'' + \lambda f = 0 & 0 \leq x \leq \ell; \\
  f'(0) = f(\ell) = 0.
\end{cases}
\]

(a) Find the eigenvalues and normalized eigenfunctions of the above RSL.

(b) Let \( S = \text{span}\{\phi_1, \phi_2\} \), the subspace of \( L^2(0, \ell) \) consisting of all possible linear combinations of the first two eigenfunctions \( \phi_1, \phi_2 \) of the above RSL. Find the best linear approximation in \( S \) to the function \( g(x) = \ell^2 - x^2 \) in the \( L^2 \) sense.

5. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= ax + y - xf(x^2 + y^2) \\
\frac{dy}{dt} &= -x + ay - yf(x^2 + y^2)
\end{align*}
\]

where \( a \) is real, \( f \) is continuous, \( f(0) = 0 \) and \( f(u) \geq u^{1/2} \).

(a) Show that the origin is the only equilibrium point and determine its linear stability.

(b) Using the Poincare-Bendixson theorem, show that there exists a stable limit cycle if \( a > 0 \).

(c) Consider the special case with \( f(u) = u^{1/2} \) for all \( r \geq 0 \) with \( a > 0 \). Find the limit cycle explicitly.

6. For the solar system, Einstein’s General Relativity can be viewed as a small perturbation to the regular Newtonian theory of gravity. The orbit of a planet going around the sun can be described in terms of the polar coordinates by \( r(\theta) \) where \( r \) is the distance from the planet to the center of mass of the system and \( \theta \) is the angle of the planet in its orbit. In General Relativity, \( r(\theta) \) is approximately governed by the equation

\[
\frac{d^2 r}{d\theta^2} + r = \frac{1}{L} + \epsilon L r^2, \quad 0 < \epsilon \ll 1
\]

where \( L \) is related to the angular momentum of the planet and \( \epsilon \) is a small positive parameter representing the deviation from the Newtonian theory. When \( \epsilon = 0 \), this is the equation for Newtonian gravity.

(a) Find the equilibrium points and classify their stability for \( \epsilon > 0 \).

(b) Find the limits of the equilibrium points as \( \epsilon \to 0 \).

(c) For \( \epsilon > 0 \) sketch the phase portrait (in the half plane \( r \geq 0 \)) and identify the region where there are periodic solutions in \( \theta \).