Problem 1. Recall the definition of the Gaussian distribution with variance $\sigma^2 > 0$:

$$p_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \ x \in \mathbb{R}.$$ 

For $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, define

$$(Tf)(x) = \sqrt{2} \int_{-\infty}^{+\infty} p_{1/2}(y)f(\sqrt{2}(x - y)) \, dy.$$ 

a) Prove that $p_1$ is a fixed point of $T$.

b) Prove that for all $c > 0$, there is exactly one fixed point of $T$ in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, say $f$, such that $\|f\|_{L^1} = c$.

c) Let $g \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ be a non-negative function. Show that the sequence $T^n g$ converges in $L^1(\mathbb{R})$ as $n \to \infty$, and find its limit.

Problem 2. Let $(t_n)_{n \geq 1}$ be a sequence of non-negative real numbers such that $\sum_{n \geq 1} t_n^{3/2} = 1$. Let $(a_n)$ be a sequence of complex numbers satisfying

$$\sum_{n \geq 1} |a_n|^3 < +\infty. \quad (1)$$ 

Define $f_n \in C([0, 1])$, by

$$f_n(x) = \sum_{m=1}^{n} t_m a_m \sin(m\pi x)$$ 

Prove that the set

$$A = \{f_n \mid n \geq 1\}$$

is precompact in $C([0, 1])$ with the supremum norm.
Problem 3.
a) Let $X^{-1}$ be the distributional limit, as $\epsilon \to 0$, of the sequence of functions
\[
F_\epsilon(x) = \begin{cases} 
\frac{1}{x}, & |x| > \epsilon \\
0, & |x| < \epsilon 
\end{cases}
\]
Show that $X^{-1}$ is the distributional derivative of the function $f(x) = \log |x|$.
b) Show that the distributional limit, as $\epsilon \to 0$, of the following sequence
\[
f_\epsilon(x) = \frac{1}{x - i\epsilon}, \quad \epsilon > 0
\]
is $X^{-1} + \pi i \delta$.

Problem 4. Let $h > 0$, and consider the following differential-difference initial-value problem, where $u(x,t)$ and $f(x)$ are $2\pi$-periodic functions of $x$:
\[
\begin{align*}
&u_t(x,t) = u(x+h,t) - 2u(x,t) + u(x-h,t), \\
&u(x,0) = f(x).
\end{align*}
\]
a) (10 points) Use Fourier series to solve for $u(x,t)$ when $f(x)$ is square-integrable.
b) (5 points) How does the smoothness of $u(\cdot, t)$ for $t > 0$ compare with the smoothness of $f(\cdot)$?
c) (5 points) Discuss briefly what happens to your solution in the limit $h \to 0$.

Problem 5.
a) (5 points) Define “orthogonal projection on a Hilbert space”.
b) (10 points) Suppose that $P$ and $Q$ are orthogonal projections with ranges $\mathcal{M}$ and $\mathcal{N}$, respectively. If $PQ = QP$, prove that $R = P + Q - PQ$ is an orthogonal projection. What is its range?

Problem 6.
a) (5 points) Define strong and weak convergence in a Hilbert space.
b) (5 points) Suppose that $(x_n)_{n=1}^\infty$ is an orthogonal sequence in a Hilbert space, meaning that $x_n$ is orthogonal to $x_m$ for $n \neq m$. Prove that the following statements are equivalent:
\[
\begin{align*}
&\text{(i) } \sum_{n=1}^\infty x_n \text{ converges strongly;} \\
&\text{(ii) } \sum_{n=1}^\infty x_n \text{ converges weakly;}
\end{align*}
\]
(iii) \( \sum_{n=1}^{\infty} \|x_n\|^2 < \infty. \)

c) (5 points) Give an example to show that if the sequence \( (x_n)_{n=1}^{\infty} \) is not orthogonal, then \( \sum_{n=1}^{\infty} x_n \) may converge weakly but not strongly.

Problem 7. Consider the system

\[
\begin{align*}
\dot{q} &= 4p^3 - 4pq \\
\dot{p} &= 2p^2 - 3q^2
\end{align*}
\]

a) Show that the function \( H(q, p) = p^4 - 2p^2q + q^3 \) is a conserved quantity for this system.

b) Compute the linearization of the system at the fixed point

\( (q^*, p^*) = \left( \frac{2}{3}, \sqrt{\frac{2}{3}} \right) \)

What type of fixed point is this? Sketch the behavior of the full system in a small neighborhood of the fixed point.

Problem 8. Consider the one-dimensional system

\[
\dot{x} = x + \frac{rx}{1 + x^2}
\]

a) Compute the location of all fixed points as a function of \( r \in \mathbb{R} \).

b) Plot the phase portrait when \( r = -2 \).

c) Plot a bifurcation diagram for the system. At what values of \( x \) and \( r \) does the bifurcation occur? What type of bifurcation is it?

d) Describe what would happen to the system’s solution if it starts at \( x = 1/2 \) and \( r = -2 \), and then \( r \) is very slowly increased? Assume that the system dynamics are much faster than the change in \( r \).