# Winter 2007: PhD Algebra Preliminary Exam

**Instructions:**

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

**Problem 1.** Let $R$ be a commutative ring with identity, and let $I$ be an ideal of $R$. Under what conditions on $I$ is $R/I$ a field? An integral domain? A commutative ring with identity?

**Problem 2.** Let $V$ be a vector space, and let $A$ and $B$ be a pair of commuting operators on $V$. Show that if $W$ is an invariant subspace for $A$, then so are the spaces $BW$ and $B^{-1}W := \{ v \in V \; : \; Bv \in W \}$.

**Problem 3.** Suppose the group $G$ has character table

$$\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
3 & -1 & 0 & \zeta_5^3+\zeta_5^2+1 & \zeta_5^4+\zeta_5+1 \\
3 & -1 & 0 & \zeta_5^4+\zeta_5+1 & \zeta_5^3+\zeta_5^2+1 \\
4 & 0 & 1 & -1 & -1 \\
5 & 1 & -1 & 0 & 0,
\end{array}$$

where $\zeta_5$ is a primitive 5-th root of unity (so $\zeta_5^4+\zeta_5^3+\zeta_5^2+\zeta_5+1=0$).

(a) Prove that $G$ is a simple group of order 60, and determine the sizes of its conjugacy classes.

(b) How does the tensor product of the two 3-dimensional irreps decompose into irreducibles?

**Problem 4.** Suppose that the group $G$ is generated by elements $x$ and $y$ that satisfy $x^5y^3 = x^8y^5 = 1$. Is $G$ the trivial group?

**Problem 5.** Let $R$ be a principal ideal domain and $I \subset R$ an ideal. Prove that every ideal in the quotient ring $R/I$ is a principal ideal. Show that $R/I$ is not necessarily a principal ideal domain.

**Problem 6.**

(a) Give an example of a $4 \times 4$ complex matrix having only one eigenvalue, equal to 3, with the space of eigenvectors having dimension 2.

(b) Let us consider the set $K$ of all matrices obeying the conditions of (a). The group $GL_4(\mathbb{C})$ acts on $K$ by means of the transformations $\phi_A(X) = AXA^{-1}$. How many orbits does this action have?

# Winter 2007: PhD Analysis Preliminary Exam

**Instructions:**

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

**Problem 1.** Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0,1]) \to C([0,1])$ be a bounded linear operator. Suppose that $\|S(p)\| \le 2$ for all polynomials $p$. Show that $S$ is the zero operator.

**Problem 2.** For $p \ge 1$, let $l^p(\mathbb{N})$ be the set of sequences $(x_n)$ such that

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

(a) Show that if $1 \le p < q < \infty$ then $l^p(\mathbb{N}) \subseteq l^q(\mathbb{N})$.

(b) Show that if $1 \le p < q < \infty$ then $l^p(\mathbb{N}) \ne l^q(\mathbb{N})$.

**Problem 3.** Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y);$$

in particular, both limits exist. Does it follow that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exists?

**Problem 4.** Let $X$ be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < Cd(x,y)$ for all $x \ne y$. The function $f$ is said to be a *weak contraction* if $d(f(x), f(y)) < d(x,y)$ for all $x \ne y$, without the constant $C$. The contraction mapping theorem says that if $f$ is a contraction, then it has a fixed point. Show that the theorem also holds when $f$ is a weak contraction and $X$ is compact.

**Problem 5.** Construct the Green's function for the Dirichlet boundary-value problem

$$-u'' + 4u = f, \quad u(0) = u(2) = 0.$$

**Problem 6.** Let $U$ be a unitary operator on a Hilbert space. Prove that the spectrum of $U$ lies on the unit circle.