Sample Graduate Preliminary Exam (post F2001 system)
(The Ph.D. Preliminary Examination is a written exam, covering graduate material in analysis and algebra, as in 201ABC and 250AB.)

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Analysis

Problem 1. a. Find a sequence of continuous functions on \([0, 1]\) converging pointwise but not uniformly.
b. Prove that \(C[0, 1]\), the space of continuous functions on \([0, 1]\), is not complete in the \(L^1\) metric
   \[ \rho(f, g) = \int |f(x) - g(x)|\,dx. \]

Problem 2. Let \(T\) be the union of the graph of \(\sin(x^{-1})\) on \((0, 1)\) and \(\{(0, 0)\}\) with the topology induced from \(R^2\). Prove that \(T\) is connected but not arcwise connected.

Problem 3. a. Prove that in any Hilbert space the parallelogram identity takes place:
   \[ ||x - y||^2 + ||x + y||^2 = 2||x||^2 + 2||y||^2. \]
b. Prove that if \(X\) is a Banach space over complex numbers such that the parallelogram identity takes place then one can make \(X\) into a Hilbert space by defining a scalar product \((x, y)\) such that \((x, y) = ||x||^2\).

Problem 4. Suppose that \(f \in L^1(\mu)\). Prove that for every \(\epsilon > 0\) there is a \(\delta > 0\) such that
   \[ \int_A |f|\,d\mu < \epsilon \quad \text{whenever} \quad \mu(A) < \delta. \]

Problem 5. State Hölder's inequality. Suppose that \(1 \leq p < q < r \leq \infty\). Prove that if \(u \in L^p \cap L^r\), then \(u \in L^q\) and
   \[ \|u\|_{L^q} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^r}^{1-\theta}, \quad \text{where} \quad \theta = \frac{1/q - 1/r}{1/p - 1/r}. \]

Problem 6. Let \(C^k([0, 1]),\ k \geq 1\), denote the set of all functions \([0, 1] \to R^1\) with a continuous \(k^{th}\) order derivative. Prove that \(C^k([0, 1])\) is dense in \(C([0, 1])\) with the supremum norm for all \(k \geq 1\).

2. Algebra

Problem 7. Suppose a group \(G\) acts on a set \(X\). Show that if \(x, y \in X\) belong to the same \(G\)-orbit, then \(|G_x| = |G_y|\) where \(G_x = \{g \in G : gx = x\}\) denotes the stabilizer of \(x \in X\).

Problem 8. Prove or give a counter example: If \(0 \to K \to G \to H \to 0\) is an exact sequence of groups with both \(K\) and \(H\) abelian, then \(G\) is abelian.

Problem 9. Prove or disprove: \(Z[x]\) is a Principle Ideal Domain.

Problem 10. Argue that the commutator subgroup of a group \(G\) is characteristic, and so is the center.

Problem 11. a. Give an example of a finite field of order 3 and a field of order 9.
b. Let \(F\) be a finite field. Show that the order of \(F\) is equal to \(p^n\) for some prime number \(p\) and a positive integer \(n\).
c. Show that the multiplicative group \(F^*\) consisting of the non-zero elements of a finite field \(F\) is a cyclic group.