PRELIMINARY EXAM IN ANALYSIS

Fall, 2015

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1.

Consider the following sequences of functions parametrized by n:

- $a_n(x) = e^{i2\sqrt{n}\pi x}, \quad x \in [0,1],$
- $b_n(x) = \sqrt{n}e^{-n|x|}, \quad x \in \mathbb{R},$
- $c_n(x) = ne^{-nx^2}, \quad x \in \mathbb{R},$
- $d_n(x) = \sum_{k=-n}^{n} e^{i2k\pi x}, \quad x \in [0,1].$

As n tends to infinity, which sequences converge (a) almost everywhere, (b) L^2-strongly, (c) L^2-weakly but not strongly. Explain your answer.

Problem 2. Let T be a linear operator on a Banach space. Show that T is bounded *if and only if* T is continuous.

Problem 3. Let $f : [0,1] \to \mathbb{R}$ be a C^1-function and suppose that $|f'(x)| \geq 1$ for all $x \in [0,1]$ and f' is monotonic. Show that

$$\left| \int_0^1 e^{i\lambda f(x)} dx \right| \leq \frac{2}{\lambda}.$$

Here i is the imaginary unit.

Problem 4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ with $f, \nabla f \in L^1(\mathbb{R}^3)$. Show that

$$\int_{\mathbb{R}^3} |f(x)|^{3/2} dx \leq \left(\int_{\mathbb{R}^3} |\nabla f(x)| dx \right)^{3/2}.$$

Problem 5. Fix a continuous function $f : [0,1] \to \mathbb{R}$. Consider the multiplication operator $M_f : C^0([0,1]) \to C^0([0,1])$ on the space $C^0([0,1])$ of continuous functions on $[0,1]$ defined by $(M_f g)(x) = f(x)g(x)$ for all $x \in [0,1]$ and $g \in C^0([0,1])$. Calculate $||M_f||$ and show that M_f is a compact operator if and only if $f \equiv 0$.

Problem 6.

Suppose that $f \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of infinitely differentiable rapidly decreasing functions

$$S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^n f^{(m)}(x)| < \infty \}$$

for all nonnegative integers $n, m = 0, 1, 2, \ldots$.

Does

$$\int_{\mathbb{R}} f(x) x^n dx = 0, \quad n = 0, 1, 2, \ldots$$

imply that f is identically zero? Explain your answer. (Hint: use the Fourier transform).

Note: $f^{(m)}$ denotes the m-th derivative of f.