ANALYSIS

Problem 1. Let $f(x,y)$ denote a C^1 function on $\mathbb{R}^2$. Suppose that

$$f(0,0)=0.$$

Prove that there exist two functions, $A(x,y)$ and $B(x,y)$, both continuous on $\mathbb{R}^2$ such that

$$f(x,y)=xA(x,y)+yB(x,y) \quad \forall (x,y)\in\mathbb{R}^2$$

(**Hint**: Consider the function $g(t)=f(tx,ty)$ and express $f(x,y)$ in terms of g via the fundamental theorem of calculus .)

Problem 2. The Fourier transform $\mathcal{F}$ of a distribution is defined via the duality relation

$$\langle \mathcal{F}f,\phi\rangle = \langle f,\mathcal{F}^*\phi\rangle$$

for all $\phi\in C_0^\infty(\mathbb{R})$, the smooth compactly-supported test functions on $\mathbb{R}$, where

$$\mathcal{F}^*\phi(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{i\xi x}\phi(\xi)d\xi\,.$$

Explicitly compute $\mathcal{F}f$ for the function

$$f(x)=\begin{cases} x, & x>0\\ 0, & x\le 0\end{cases}.$$

Problem 3. Let $\{P_n(x)\}_{n=1}^\infty$ denote a sequence of polynomials on $\mathbb{R}$ such that

$$P_n\to 0 \text{ uniformly on } \mathbb{R} \text{ as } n\to\infty.$$

Prove that, for n sufficiently large, all P_n are constant polynomials.

Problem 4. For $g\in L^1(\mathbb{R}^3)$, the convolution operator G is defined on $L^2(\mathbb{R}^3)$ by

$$Gf(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} g(x-y)f(y)dy\,, \quad f\in L^2(\mathbb{R}^3)\,.$$

Prove that the operator G with

$$g(x)=\frac{1}{4\pi}\frac{e^{-|x|}}{|x|}\,, \quad x\in\mathbb{R}^3,$$

is a bounded operator on $L^2(\mathbb{R}^3)$, and the operator norm $\|G\|_{op}\le 1$.

Problem 5. Consider the map which associates to each sequence $\{x_n\,:\, n\in\mathbb{N}\,,\, x_n\in\mathbb{R}\}$ the sequence, $\{(F(\{x_n\}))_m\,;m\in\mathbb{N}\,,\,(F(\{x_n\}))_m\in\mathbb{R}\}$, defined as follows:

$$\Big\{F(\{x_n\})\Big\}_m := \frac{x_m}{m} \quad \text{for} \quad m=1,2,\ldots$$

1. (1) Determine (with proof) the values of $p\in[1,\infty]$ for which the map $F: l^p\to l^1$ is well-defined and continuous.
2. (2) Next, determine the values of $q\in[1,\infty]$ for which the map $F: l^q\to l^2$ is well-defined and continuous.

Note for $1 \leq p < \infty$, l^p denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, while l^∞ denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sup_{n \in \mathbb{N}} |x_n| < \infty$.

Problem 6. For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof, e.g. by citing a relevant theorem; if false, give a counterexample.

Let $\mathbb{H}$ denote a separable Hilbert space and (x_n) a sequence of $\mathbb{H}$.

(a) If (x_n) is weakly convergent then it is strongly convergent.

(b) If (x_n) is strongly convergent then it is bounded.

(c) If (x_n) is weakly convergent then it is bounded.

(d) If (x_n) is bounded, there exists a strongly convergent subsequence of (x_n).

(e) If (x_n) is bounded, there exists a weakly convergent subsequence of (x_n).

(f) If (x_n) is weakly convergent and T is a bounded linear operator from $\mathbb{H}$ to $\mathbb{R}^d$, for some d, then $T(x_n)$ converges in $\mathbb{R}^d$.