MA Algebra Preliminary Exam for 2004-05

Instructions: All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

Problem 1. Let $F$ be a field, $n,m$ be positive integers and $A$ be an $n \times n$ matrix with coefficients in $F$. Suppose that $A^m = 0$. Show that $A^n = 0$.

Problem 2. Prove that $f(x) = x^4 + x + 1$ is irreducible over $\mathbb{Q}$.

Problem 3. Prove that $\mathbb{Q}$ contains no proper subgroups of finite index.

Problem 4. Give definition of PID and examples (without proofs) of:
   a. A commutative ring which is a PID.
   b. A commutative ring which is not a PID.

Problem 5. Let $p$ be a prime number and $G = \mathbb{Z}_p$ be the finite cyclic group of order $p$. Prove that the group of automorphisms of $G$ is cyclic and compute its order.

Problem 6. Let $S_n$ denote the group of permutations of $n$ objects. Find four different subgroups of $S_4$ isomorphic to $S_3$ and nine isomorphic to $S_2$. 
PhD Algebra Preliminary Exam for 2004-05

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Problem 1. Let $F$ be a field, $n, m$ be positive integers and $A$ be an $n \times n$ matrix with coefficients in $F$. Suppose that $A^m = 0$. Show that $A^n = 0$.

Problem 2. Prove that $f(x) = x^4 + x + 1$ is irreducible over $\mathbb{Q}$.

Problem 3. Prove that $\mathbb{Q}$ contains no proper subgroups of finite index.

Problem 4. Let $F$ be a functor from the category of sets into the category of sets. Prove that if for some non-empty set $X$, the set $F(X)$ is empty, then the set $F(Y)$ is empty for every set $Y$.

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Analysis Preliminary Exam for 2004-05

Instructions: All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

Problem 1. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \pi/2 + x - \arctan(x)$$

has no fixed points in $\mathbb{R}$ and that

$$|T(x) - T(y)| < |x - y|, \text{ for all distinct } x, y \in \mathbb{R}$$

Why does not this example contradict the contraction mapping theorem?

Problem 2. Prove that the vector space $C([a, b])$ is separable.

Here and below, $C([a, b])$ is the vector space of continuous functions $f : [a, b] \to \mathbb{R}$ with the supremum norm.

Problem 3. Suppose that $f_n \in C([a, b])$ is a sequence of functions converging uniformly to a function $f$. Show that

$$\lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

Give a counterexample to show that the pointwise convergence of continuous functions $f_n$ to a continuous function $f$ does not imply convergence of the corresponding integrals.

Problem 4. Let $\ell_2(\mathbb{Z})$ denote the complex Hilbert space of sequences $x_n \in \mathbb{C}, n \in \mathbb{Z}$, such that

$$\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty.$$ 

Define the shift operator $S : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by

$$S((x_n)) = (x_{n+1}).$$

Show that $S$ has no eigenvalues.

Problem 5. Consider the initial value problem

$$u'(t) = |u(t)|^\alpha, u(0) = 0.$$
Show that the solution of this problem is unique if $\alpha > 1$ and is not unique if $0 \leq \alpha < 1$.

**Problem 6.** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H}_0$ a dense linear subspace of $\mathcal{H}$, $(x_n)$ a sequence in $\mathcal{H}$ and $x \in \mathcal{H}$ such that (i) there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n$, (ii) $\lim_n \langle x_n, y \rangle = \langle x, y \rangle$, for all $y \in \mathcal{H}_0$. Prove that $(x_n)$ converges to $x$ in the weak topology of $\mathcal{H}$. 