Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
January 2, 2009

Instructions:

• This exam has 3 pages (8 problems) and is closed book.

• The first 6 problems cover Analysis and the last 2 problems cover ODEs.

• Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

• Use separate sheets for the solution of each problem.

Problem 1: (10 points)
Let $1 < p < 2$.

(a) Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.

(b) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$.

Problem 2: (10 points)

(a) State the Weierstrass approximation theorem.

(b) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x) \, dx = 0$$

for all non-negative integers $n$. Prove that $f = 0$.

Problem 3: (10 points)

(a) Define strong convergence, $x_n \to x$, and weak convergence, $x_n \rightharpoonup x$, of a sequence $(x_n)$ in a Hilbert space $\mathcal{H}$.

(b) If $x_n \rightharpoonup x$ weakly in $\mathcal{H}$ and $\|x_n\| \to \|x\|$, prove that $x_n \to x$ strongly.
(c) Give an example of a Hilbert space $\mathcal{H}$ and sequence $(x_n)$ in $\mathcal{H}$ such that $x_n \rightharpoonup x$ weakly and

$$\|x\| < \liminf_{n \to \infty} \|x_n\|.$$ 

**Problem 4:** (10 points)

Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a complex Hilbert space $\mathcal{H}$ such that

$$T^* = -T, \quad T^2 = -I$$

and $T \neq \pm iI$. Define

$$P = \frac{1}{2} (I + iT), \quad Q = \frac{1}{2} (I - iT).$$

(a) Prove that $P, Q$ are orthogonal projections on $\mathcal{H}$.

(b) Determine the spectrum of $T$, and classify it.

**Problem 5:** (10 points)

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(Hf)(\xi) = \text{isgn}(\xi) \hat{f}(\xi) = \begin{cases} i\hat{f}(\xi) & \text{if } \xi > 0, \\ -i\hat{f}(\xi) & \text{if } \xi < 0, \end{cases}$$

where $\hat{f}$ denotes the Fourier transform of $f$.

(a) Why is $Hf \in L^2(\mathbb{R})$ for any $f \in \mathcal{S}(\mathbb{R})$?

(b) If $f \in \mathcal{S}(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x) \, dx = 0.$$ 

[Hint: you may want to use the Riemann-Lebesgue Lemma.]

**Problem 6:** (10 points)

Let $\Delta$ denote the Laplace operator in $\mathbb{R}^3$.

(a) Prove that

$$\lim_{\epsilon \to 0} \int_{B_\epsilon^c} \frac{1}{|x|} \Delta f(x) \, dx = 4\pi f(0), \quad \forall f \in \mathcal{S}(\mathbb{R}^3)$$

where $B_\epsilon^c$ is the complement of the ball of radius $\epsilon$ centered at the origin.

(b) Find the solution $u$ of the Poisson problem

$$\Delta u = 4\pi f(x), \quad \lim_{|x| \to \infty} u(x) = 0$$

for $f \in \mathcal{S}(\mathbb{R}^3)$.
Problem 7: (8 points)
Show that the solution to the system

$$\dot{x} = 1 + x^{10}$$

goes to infinity in finite time.

Problem 8: (12 points)
Consider the nonlinear system of ODEs:

$$\begin{align*}
\dot{x} &= y - x \left( (x^2 + y^2)^4 - \mu \left( (x^2 + y^2)^2 - 1 \right) - 1 \right) \\
\dot{y} &= -x - y \left( (x^2 + y^2)^4 - \mu \left( (x^2 + y^2)^2 - 1 \right) - 1 \right)
\end{align*}$$

(a) Rewrite the system in polar coordinates.

(b) For $0 \leq \mu < 1$, show that the circular region that lies within concentric circles with radius $r_{min} = 1/2$ and $r_{max} = 2$ is a trapping region. And use the Poincaré-Bendixson theorem to show that there exists a stable limit cycle.

(c) Show that a sub-critical Hopf Bifurcation occurs at $\mu = 1$. 