MA Algebra Preliminary Exam for 2005-06

Instructions: All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

Problem 1. Let field $E$ be a finite extension of a field $F$, and let $R$ be a subring of $E$ that contains $F$. Prove that $R$ is a field.

Problem 2. Dihedral group $D_n$ is defined as a group of rigid motions on the Euclidean plane $\mathbb{R}^2$ preserving a regular $n$-gon in $\mathbb{R}^2$. Check that there exists a system of generators of this group consisting of two elements. Describe all endomorphisms (i.e. homomorphisms $D_n \to D_n$) of the group $D_n$. Calculate the number of endomorphisms (as a function of $n$).

Problem 3. Describe all possible Jordan forms of an $n \times n$ matrix $X$ obeying $X^n = 0$.

Problem 4. Show that $\mathbb{Q}$ (the additive group of rational numbers) is not finitely generated.

Problem 5. Determine all finitely generated abelian groups which have finite group of automorphisms.

Problem 6. Suppose that $H \subset G$ is a subgroup which is contained in every nontrivial subgroup of $G$. Show that $H$ is contained in the center of $G$. 