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Senior Thesis

Lieb-Robinson Bounds on 1D Lattices

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Abstract

We explore the locality of time-evolved observables on finite spin-chains. Initially, we examine the derivation of the general Lieb-Robinson bound for a one-dimensional spin chain and then develop an improved bound in the case of short-range interactions. In addition, we study systems of coupled chains with known Lieb-Robinson velocities and demonstrate improved locality estimates for weakly and strongly coupled chains which hold for meaningful time regimes.

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1 Introduction

In physics, locality is a fundamental attribute of any physical system. Indeed, the theory of relativity is built on the axiom that nothing, whether matter or information, can travel at greater speeds than light. This property implies localization in the sense that objects at a distance act independently of each other, at least for short periods of time. In non-relativistic mechanics, however, it is thought that there is no bound limiting the speed at which information can propagate, and in quantum mechanics it is often preferable to use non-relativistic models to describe a system, such as in Hamiltonian dynamics.

For quantum systems consisting of atoms arranged in a lattice, it can be shown that despite its non-relativistic structure the system still exhibits locality. This property was first mathematically demonstrated by Lieb and Robinson in 1972 [4], where they showed a theoretical upper limit on the speed which information propagates through finite spin systems by bounding the spread of local observables. The bound was generalized by in 2006 by Nachtergaele and Sims [6] to a variety of quantum systems defined on sets of vertices with a given metric. The Lieb-Robinson bound also has applications to the exponential decay of correlations [5, 6], and in 2012 the bound was experimentally observed by Cheneau et al. [1].

We study this Lieb-Robinson bound and its associated propagation velocity (known as the Lieb-Robinson velocity) for a one-dimension spin system, often referred to as a spin-chain. In section 1.1, we lay out the framework of a spin chain and discuss the relevant representations of quantum dynamics in sections 1.2–1.4. After this, we seek to improve on Lieb's and Robinson's original work in specific cases. Following principles laid out in [7], we derive the general Lieb-Robinson bound in section 2. We then discuss improvements for systems with short-range interactions in section 2.5.

After establishing these general localizing properties about the spin system, we consider more specific cases. For certain spin chains, it is reasonable to assume additional information about the interactions at certain sites. In section 3, we study the behavior of systems composed of coupling several well-understood chains together. Specifically, we show that weakly coupled systems demonstrate greater localization for short times and develop a bound which improves on the standard Lieb-Robinson in certain strongly coupled chains.

1.1 Spin Chains

In order to study the dynamics of a quantum spin system, we recall the basic mathematical structure used to describe quantum mechanics. Traditionally, particles or systems of particles are described by a wave function, typically denoted by $\Psi(\mathbf{x}, t)$ which satisfies Schrödinger's equation. In a more abstract sense, the state of a particle determined by Ψ is viewed as vector in some well defined space, typically a complex Hilbert space (a complete inner product space). Linear transformations defined on this Hilbert space are functions which take in a state and return some new state. Additionally, for any system each measurable physical quantity—such as the energy, position or momentum of a particle—is represented by a linear transformation known as an observable. These observables are represented by bounded, self-adjoint operators on the Hilbert space of states, and together these observables generate a C^*-algebra $\mathcal{A}$ that is referred to as an *algebra of observables*. A brief overview of the three pictures describing quantum mechanics is given beginning in section 1.2, however we first lay out the setup of a lattice system below.

In more precise terms, for a finite spin chain of length N we let $\Lambda = \{1, \ldots, N\}$ be an integer lattice and define the Hilbert space for the system by

$$\mathcal{H} = \otimes_{j\in\Lambda}\mathbb{C}^{d_j} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_N}.$$

Here, $\otimes$ represents the tensor product, and $\mathbb{C}^{d_i}$ is a d_i-dimensional complex Hilbert space. Given $\mathcal{H}$, we denote an algebra of observables as the set of bounded linear operators ($\mathcal{B}(\mathcal{H})$) on $\mathcal{H}$. This algebra is denoted

$$\mathcal{A} = \mathcal{B}(\mathcal{H}) = \otimes_{j\in\Lambda} M_{d_j}(\mathbb{C}).$$

For subsets $X \subset \Lambda$, the algebra on X is given by $\mathcal{A}_X = \otimes_{j\in X} M_{d_j}(\mathbb{C})$ which has a natural inclusion from $\mathcal{A}_X \to \mathcal{A}_\Lambda$ by $\otimes_{j\in X} P_j \mapsto \otimes_{j\in\Lambda} Q_j$, where

$$Q_j = \begin{cases} P_j, & \text{if } j \in X \\ \mathbb{1}_{d_j}, & \text{if } j \in \Lambda \setminus X. \end{cases}$$

Thus, by identifying $A \in \mathcal{A}_X$ by its inclusion to $\mathcal{A}_\Lambda$, we can say $\mathcal{A}_X \subseteq \mathcal{A}_\Lambda$. Further, under this identification, we say the support of an observable A is the minimal set S such that there exists $A' \in \mathcal{A}_S$ and $A = A' \otimes_{j\in\Lambda\setminus S} \mathbb{1}$.

For the systems we are interested in, we also must define the Hamiltonian for the chain. A Hamiltonian is a self-adjoint operater determining the time evolution of state vectors arising in the Schrödinger picture (more to be said on this in section 1.2). In the context of spin chains, the Hamiltonian on the lattice Λ is determined by the sum of interactions that act on the power set $\mathcal{P}(\Lambda)$. An interaction Φ is a function from $\mathcal{P}(\Lambda)$ to $\mathcal{A}_\Lambda$ with the property that for every $Z \subset \Lambda$, $\Phi(Z) \in \mathcal{A}_Z$ and $\Phi(Z)^* = \Phi(Z)$.

1.2 Schrödinger Picture

In the Schrödinger picture, the states are viewed as time dependent, and evolve in time by a unitary operator in the Hilbert space $\mathcal{H}$. If Ψ_0 is some state, then we would say that the time evolved state is $\Psi(t) = U(t)[\Psi_0]$, where $U(t)$ is a unitary time evolution operator in $\mathcal{H}$. This $\Psi(t)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = H\Psi(t),$$

where H is the Hamiltonian, an observable representing the possible outcomes from measuring the energy of the system. Noting that $\Psi(t) = U(t)\Psi_0$, we obtain from Schrödinger's equation that

$$\frac{\partial U(t)}{\partial t} = -\frac{iH}{\hbar}U(t).$$

This is easily solved, giving the time evolution operator $U(t) = e^{-itH/\hbar}$, where $e^{-itH/\hbar}$ is defined by the series $\mathbb{1} - it/\hbar + \frac{1}{2!}(itH/\hbar)^2 - \frac{1}{3!}(itH/\hbar)^3 + \cdots$. It is easy to confirm that $U(t)$ is indeed unitary, since $U(t)U^*(t) = e^{-itH/\hbar}e^{itH/\hbar} = \mathbb{1}$. This guarantees that the norm of the state $\|\Psi\|$ remains constant through time. In this same vein, we define for any observable A in the set of bounded linear operators on $\mathcal{H}$ (denoted $\mathcal{B}(\mathcal{H})$), the expectation of A by

$$\langle\Psi(t), A\Psi(t)\rangle.$$

While the Schrödinger picture focuses on the state of the system, an alternate view of the quantum system is to consider the dynamics of the observable on $\mathcal{H}$, as is done in the Heisenberg picture.

1.3 Heisenberg Picture

Unlike in Schrödinger's picture, Heisenberg concentrates on the observable A. Starting from Schrödinger's picture, we know the expectation of A is given by

$$\begin{aligned}\langle\Psi(t), A\Psi(t)\rangle =& \langle e^{-itH/\hbar}\Psi_0, Ae^{-itH/\hbar}\Psi_0\rangle \\ =& \langle\Psi_0, e^{itH/\hbar}Ae^{-itH/\hbar}\Psi_0\rangle.\end{aligned}$$

We can think of the expression $e^{itH/\hbar}Ae^{-itH/\hbar}$ as a time evolved observable formed by conjugation of A by the unitary operator $e^{itH/\hbar}$. This time evolved observable will be a main area of focus for the remainder of this paper. For notational ease, we will absorb Planck's constant $\hbar$ into the Hamiltonian H, and denote the time evolution $e^{itH}Ae^{-itH}$ by $\tau_t(A)$.

1.4 Interaction Picture

The last picture of quantum mechanics we will consider is the interaction (Dirac) picture, which is a useful method when analyzing Heisenberg dynamics. The interaction picture blends both Schrödinger and Heisenberg pictures together and views either of the quantum states and operators as time dependent. It is especially useful for considering problems where the Hamiltonian H is composed of the sum of two terms $H = H_0 + V_I$, where H_0 is well understood and exactly solvable while V_I is some perturbation interaction that is more difficult to analyze. This is common when the Hamiltonian has time dependent terms, and it is advantageous to group these with the perturbation term V_I.

Under this picture, the perturbation interaction is transformed like Heisenberg operators as such

$$V_I(t) = e^{itH_0}V_Ie^{-itH_0}.$$

From this we have that the time dependent Hamiltonian operator under Heisenberg dynamics is $e^{itH_0}(H_0 + V_I)e^{-itH_0} = H_0 + V_I(t)$, since H_0 commutes with itself in the series expansion of the exponential. So, the time dependent portion of H is determined solely by the perturbation term. If the interaction state Ψ_I transformed by the free portion of the Hamiltonian H_0 is called

$$\Psi_I(t) = e^{itH_0}\Psi(t),$$

where $\Psi(t)$ is the Schrödinger state vector $e^{-itH}\Psi_0$, then we see that

$$\begin{aligned}
\frac{d}{dt}\Psi_I(t) &= iH_0\Psi_I(t) + e^{itH_0}\frac{d}{dt}\Psi(t) \\
&= iH_0\Psi_I(t) + e^{itH_0}\left(-iH\Psi(t)\right) \\
&= iH_0\Psi_I(t) - ie^{itH_0}(H_0 + V_I)\Psi(t) \\
&= -ie^{itH_0}V_I\Psi(t) \\
&= -ie^{itH_0}V_I\left(e^{-itH_0}e^{itH_0}\right)\Psi(t) \\
&= -iV_I(t)\Psi_I(t).
\end{aligned}$$

So, the time evolution of the state vectors in the interaction picture depends only on the perturbation term.

If we suppose that $\Psi_I(t) = U(t)\Psi_0$, where $U(t)$ is some unitary operator as in the Schrödinger picture, then from the computation of $\frac{d}{dt}\Psi_I(t)$ above, we see that the operator U satisfies $\frac{d}{dt}U(t) = -iV_I(t)U(t)$. Then, because $U(t)\Psi_0 = \Psi_I(t) = e^{itH_0}\Psi(t)$, it follows that $e^{-itH_0}U(t) = e^{-itH}$ which implies the full dynamics controlled by Hamiltonian $H = H_0 + V_I$ is determined entirely by the free dynamics from H_0 and the unitary $U(t)$.

When studying the time evolution of observables, we see that τ_t can then be written as the composition $\tau_t^0 \circ \tau_t^I$, where τ_t^0 is conjugation by e^{itH_0} and τ_t^I is conjugation by $U(t)$. This property proves useful in section 3 when studying coupled spin systems.

2 Lieb Robinson Bound

Lieb and Robinson's original work [4] initially demonstrated the existence of a finite information propagation speed throughout a quantum spin system. Provided $\Gamma \subset \mathbb{Z}^d$ is some d-dimensional lattice on which a bounded and translation invariant interaction is defined, then given any observables A and B with finite supports $X \subset \Gamma$ and $Y \subset \Gamma$, respectively, Lieb and Robinson showed that there exists a finite v such that for all times $t \in \mathbb{R}$,

$$\|[\tau_t(A), B]\| \leq ce^{-\alpha(d(X,Y)-v|t|)}.$$

Here, $d(X,Y)$ is the distance between the sets X and Y, $a > 0$, and c is some positive constant that depends on the size of the supports of A and B, the interaction, lattice structure, and the dimension of the Hilbert space on which the system is defined. The value v is called the Lieb-Robinson velocity and it gives a "light-cone" of sorts since the further outside of the region $d(X,Y) - v|t| < 0$, the greater the exponential decay becomes. Since this time, improvements to this bound have removed the assumptions of bounded or translation invariant interactions [3, 5]. Additionally, [8] removes the dependence of the prefactor c on the lattice structure or on single site Hilbert space dimensions. Independence of Hilbert space dimension, in particular, is significant as it allows generalizing the bound to infinite Hilbert spaces [9].

2.1 Setup

We consider a one-dimensional spin system and work to develop various Lieb-Robinson type bounds on it. Here, we present several preliminary notational conventions and remarks which will simplify the proofs and discussion throughout this paper. In considering the dynamics on a lattice $\Lambda \subset \mathbb{Z}$, given a subset $X \subset \Lambda$ and an interaction Φ, we must identify where there exists non-zero interactions between X and X^c.

Definition 1. Given $\Lambda \subseteq \mathbb{Z}$ and $X \subset \Lambda$, we define the surface of X as those sets in $Z \in \mathbb{Z}$ which satisfy both:

i. $Z \cap X$ and $Z \cap X^c$

ii. $\Phi(Z) \neq \emptyset$.

We denote the set of all such Z by $S_\Lambda(X)$.

Given this notion of surface, we further define the Φ-boundary of a set X as those lattice points in X which make up the surface of X.

Definition 2. Given an interaction Φ and $X \subset \Lambda$, we define the $\Phi-$boundary of X by $\partial_\Phi(X) = \{x \in X : \exists\ Z \in S_\Lambda(X) \text{ with } x \in Z \text{ and } \Phi(Z) \neq 0\}$

Notice that $S_\Lambda(X)$ is a set consisting of subsets of Λ, while $\partial_\Phi(X)$ is a subset of lattice points from X. We also note the following properties of τ_t^Λ, the Heisenberg time evolution operator on the lattice Λ.

Proposition 1. *The operator τ_t^Λ is an isometric $*$-algebra homomorphism from $\mathcal{A}_\Lambda$ to $\mathcal{A}_\Lambda$. That is for all $A, C \in \mathcal{A}_{Z\subset\Lambda}$ and $\lambda \in \mathbb{C}$, the following properties hold:*

i) $\tau_t^\Lambda(\lambda A + C) = \lambda\tau_t(A) + \tau_t(C)$

ii) $\tau_t^\Lambda(AC) = \tau_t^\Lambda(A)\tau_t^\Lambda(C)$.

iii) $\tau_t^\Lambda(A)^* = \tau_t^\Lambda(A^*)$

iv) $\|\tau_t^\Lambda(A)\| = \|A\|$, *where* $\|\cdot\|$ *is the operator norm.*

Proof. Let $A, C \in \mathcal{A}_Z$ and $\lambda \in \mathbb{C}$.

i) Since, $\tau_t^\Lambda(A) = e^{itH_\Lambda}Ae^{-itH_\Lambda}$, we have

$$\begin{aligned}\tau_t^\Lambda(\lambda A + C) &= e^{itH_\Lambda}(\lambda A + C)e^{-itH_\Lambda} = e^{itH_\Lambda}\lambda Ae^{-itH_\Lambda} + e^{itH_\Lambda}Ce^{-itH_\Lambda}\\ &= \lambda e^{itH_\Lambda}Ae^{-itH_\Lambda} + e^{itH_\Lambda}Ce^{-itH_\Lambda} = \lambda\tau_t(A) + \tau_t(C)\end{aligned}$$

ii)

$$\begin{aligned}\tau_t^\Lambda(AC) &= e^{itH_\Lambda}ACe^{-itH_\Lambda} = e^{itH_\Lambda}A\cdot\mathbb{1}\cdot Ce^{-itH_\Lambda}\\ &= e^{itH_\Lambda}A(e^{-itH_\Lambda}e^{itH_\Lambda})Ce^{-itH_\Lambda} = \tau_t^\Lambda(A)\tau_t^\Lambda(C)\end{aligned}$$

iii)

$$\begin{aligned}\tau_t^\Lambda(A)^* &= \left(e^{itH_\Lambda}Ae^{-itH_\Lambda}\right)^* = \left(Ae^{-itH_\Lambda}\right)^*\left(e^{itH_\Lambda}\right)^*\\ &= e^{itH_\Lambda}A^*e^{-itH_\Lambda} = \tau_t^\Lambda(A^*)\end{aligned}$$

iv) Since H_Λ is self adjoint, we have that e^{itH_Λ} and e^{-itH_Λ} are unitary. Thus,

$$\|e^{itH_\Lambda}Ae^{-itH_\Lambda}\| = \|Ae^{-itH_\Lambda}\| = \|A\|,$$

so $\|\tau_t^\Lambda(A)\| = \|A\|$.

□

Proposition 2 (Jacobi Identity)**.** *The commutator satisfies Jacobi's identity*

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0.$$

Proof. This is verified quickly by expanding the commutators.

$$\begin{aligned}&[A,[B,C]] + [B,[C,A]] + [C,[A,B]]\\ &= A[B,C] - [B,C]A + B[C,A] - [C,A]B + C[A,B] - [A,B]C\\ &= ABC - ACB - BCA + CBA + BCA - BAC -\\ &\qquad CAB + ACB + CAB - CBA - ABC + BAC\\ &= (ABC - ABC) + (ACB - ACB) + (BCA - BCA) +\\ &\qquad (CBA - CBA) + (BAC - BAC) + (CAB - CAB)\\ &= 0\end{aligned}$$

□

In Lemma 4, we make use of a *norm preserving* property, which we define here.

Definition 3. Let X be a finite dimensional normed space with operator norm $\|\cdot\|$, and let $I \subseteq \mathbb{R}$. Suppose $A : I \to \mathcal{L}(X)$ is a continuous function with range $\mathcal{L}(X)$, the linear operators on X. Let $x(t)$ be the solution to the differential equation

$$\partial_t x(t) = A(t)x(t)$$

with initial condition $x(t_0) = x_0 \in X$. Then, we say that $A(t)$ is *norm preserving* if for every $x_0 \in X$, the solution operator $T : X \to X$ given by $T(t)[x_0] = x(t)$ satisfies

$$\|T(t)[x_0]\| = \|x_0\|$$

We will also make use of an $\mathfrak{F}$-function in order to bound the decay of interactions separated by some distance. Below, we define such a function and its properties.

Definition 4. An $\mathfrak{F}$-function $F : [0, \infty) \to (0, \infty)$ satisfies the following

(i) F is nonincreasing

(ii) F is *uniformly integrable*, in the sense that $\|F\| := \sup_{z\in\mathbb{Z}} \left\{ \sum_{y\in\mathbb{Z}} F(|z-y|) \right\} < \infty$

(iii) There exists a constant $C_F > 0$ such that $\sum_{z\in\mathbb{Z}} F(|x-z|)F(|z-y|) \le C_F F(|x-y|)$

and we denote C_F the convolution constant.

Given an $\mathfrak{F}$-function F and an interaction Φ, we define the $\mathfrak{F}$-norm of Φ by

$$\|\Phi\|_F := \sup_{x,y\in\mathbb{Z}} \left\{ \sum_{X \ni x,y} \frac{\|\Phi(X)\|}{F(|x-y|)} \right\}.$$

With this norm, we can formally define a bounded interaction Φ as one which satisfies $\|\Phi\|_F < \infty$ and identify the set $\mathcal{B}_F(\Lambda)$ to be the set of interactions with finite $\mathfrak{F}$-norm. It is also worth noting that given any $\mathfrak{F}$-function F, defining $F_a(r) = e^{-ar}F(r) \le F(r)$ with $a \ge 0$ gives another $\mathfrak{F}$-function with convolution constant $C_{F_a} \le C_F$.
Some examples of functions that satisfy Definition 4 are $F(r) = \frac{1}{1+|r|^\beta}$ and $F_a(r) = \frac{e^{-ar}}{1+|r|^\beta}$, where $\beta > 1$.

2.2 General Bound

To obtain a general bound on propagation through a lattice, we consider a 1D system represented by the interval $\Lambda = [a, b] \subseteq \mathbb{Z}$ and let X and Y be disjoint intervals on Λ so that $X, Y \subset \Lambda$ and $X \cap Y = \emptyset$. Further, suppose the Hamiltonian, denoted H_Λ, is derived from a bounded interaction $\Phi \in \mathcal{B}_F(\Lambda)$, for some $\mathfrak{F}$-function F. Finally, we consider $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, observables supported on the algebras of X and Y, respectively. Using the Heisenberg picture, we study the time evolution of A on Λ by $\tau_t^\Lambda(A) = e^{itH_\Lambda}Ae^{-itH_\Lambda}$, and consider how the support of A spreads with time by analyzing the norm of the commutator $[\tau_t^\Lambda(A), B]$.

Theorem 3 (Lieb-Robinson Bound)**.** *Let Λ be a finite subset of $\mathbb{Z}$, and let H_Λ be a Hamiltonian determined by the interaction Φ over Λ so that $H_\Lambda = \sum_{Z\subset\Lambda} \Phi(Z)$. Let X and Y be subsets of Λ, with $X \cap Y = \emptyset$ and define $\mathcal{A}_X$ (resp. $\mathcal{A}_Y$) as the algebra of local observables over X (resp. Y). If A and B are any observables such that $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, then we may estimate*

$$\|[\tau_t^\Lambda(A), B]\| \le \frac{2\|A\|\|B\|}{C_F} \left(e^{2\|\Phi\|_F C_F |t|} - 1 \right) D(X, Y)$$

for all $t \in \mathbb{R}$.
Here, C_F *is the convolution constant for the* $\mathfrak{F}$*-function used to determine* $\|\Phi\|_F$*, and* $D(X,Y)$ *is defined by*

$$D(X,Y) = \min\left\{ \sum_{x\in\partial_\Phi X}\sum_{y\in Y} F(d(x,y)), \sum_{x\in X}\sum_{y\in\partial_\Phi Y} F(d(x,y)) \right\}$$

2.3 An integral bound for $\|[\tau_t^\Lambda(A), B]\|$

To prove Theorem 3, we begin by studying the behavior of $[\tau_t^\Lambda(A), B]$. It is useful to split the dynamics of τ^Λ into forward and backward propagation and consider $f(t) = [\tau_t^\Lambda(\tau_{-t}^X(A)), B]$, where $\tau^X(\cdot)$ is the Heisenberg dynamics generated by Hamiltonian $H_X = \sum_{Z\subset X}\Phi(Z)$. Since we are interested in how f changes with time, it is logical to study its time derivative. First we note that $\frac{d\tau_t^\Lambda(\cdot)}{dt} = [iH_\Lambda, \tau_t^\Lambda(\cdot)]$, so using the product rule we obtain

$$\begin{aligned}\frac{d\tau_t^\Lambda(\tau_{-t}^X(A))}{dt} =& iH_\Lambda e^{itH_\Lambda}\tau_{-t}^X(A)e^{-itH_\Lambda} + e^{itH_\Lambda}\left(-[iH_X, \tau_{-t}^X(A)]e^{-itH_\Lambda} - \tau_{-t}^X(A)e^{-itH_\Lambda}iH_\Lambda\right)\\ =&\tau_t^\Lambda(iH_\Lambda\tau_{-t}^X(A)) + \tau_t^\Lambda\left(-[iH_X, \tau_{-t}^X(A)] - \tau_{-t}^X(A)iH_\Lambda\right) \qquad (2.1)\\ =& i\tau_t^\Lambda([H_\Lambda, \tau_{-t}^X(A)] - [H_X, \tau_{-t}^X(A)]) \qquad (2.2)\\ =& i\tau_t^\Lambda([H_\Lambda - H_X, \tau_{-t}^X(A)]) \qquad (2.3)\end{aligned}$$

where (2.1) holds since H_Λ and $e^{\pm itH_\Lambda}$ commute and (2.2) and (2.3) hold by the linearity of $\tau(\cdot)$ and $[\cdot,\cdot]$.

We then notice that $H_\Lambda - H_X$ can be expressed in terms of sets in $S_\Lambda(X)$ by writing

$$\mathcal{P}(\Lambda) = \{Z : Z \subseteq X^c\} \cup S_\Lambda(X) \cup \{Z : Z \subseteq X\}.$$

So

$$H_\Lambda - H_X = \sum_{Z\subset\Lambda}\Phi(Z) - \sum_{Z\subseteq X}\Phi(Z) = \sum_{Z\subseteq X^c}\Phi(Z) + \sum_{Z\in S_\Lambda(X)}\Phi(Z). \qquad (2.4)$$

Now, Eq. (2.4) demonstrates the Hamiltonian difference divides into two sums, and furthermore, the first term in the sum is supported entirely outside the support of A. Therefore, $\sum_{Z\subseteq X^c}\Phi(Z)$ will commute with $\tau_{-t}^X(A)$ allowing Eq. (2.3) to be rewritten as

$$\frac{d\tau_t^\Lambda(\tau_{-t}^X(A))}{dt} = i\sum_{Z\in S_\Lambda(X)}\tau_t^\Lambda([\Phi(Z), \tau_{-t}^X(A)]) = i\sum_{Z\in S_\Lambda(X)}[\tau_t^\Lambda(\Phi(Z)), \tau_t^\Lambda(\tau_{-t}^X(A))] \qquad (2.5)$$

where the second equality follows by the bi-linearity of the commutator and since τ is homomorphic under multiplication and addition. Combining the results from equations (2.3) and (2.5), we arrive at

$$\begin{aligned}f'(t) &= [\frac{d\tau_t^\Lambda(\tau_{-t}^X(A))}{dt}, B]\\ &= i\left[[\tau_t^\Lambda([H_\Lambda - H_X, \tau_{-t}^X(A)]), B\right]\\ &= i\sum_{Z\in S_\Lambda(X)}[[\tau_t^\Lambda(\Phi(Z)), \tau_t^\Lambda(\tau_{-t}^X(A))], B] \qquad (2.6)\\ &= i\sum_{Z\in S_\Lambda(X)}[\tau_t^\Lambda(\Phi(Z)), f(t)] - i\sum_{Z\in S_\Lambda(X)}[\tau_t^\Lambda(\tau_{-t}^X(A)), [\tau_t^\Lambda(\Phi(Z)), B]]\end{aligned}$$

where the last equality follows from rearranging the nested commutator using Jacobi's identity (Prop. 2).

With $f(t)$ expressed in a differential equation, we employ the following lemma, proved in A.1, to bound the solution to the differential equation (2.6).

Lemma 4. *Let $A(t)$, for $t \in I \subset \mathbb{R}$, be a family of norm preserving operators on a finite dimensional normed space X. For any continuous function $b : I \to X$, the solution of,*

$$\partial_t y(t) = A(t)y(t) + b(t) \tag{2.7}$$

with initial condition $y(t_0) = y_0$, satisfies the bound

$$\|y(t) - T(t)(y_0)\| \leq \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \|b(s)\| ds \tag{2.8}$$

where T is the solution operator such that $f(t) = T(t)f_0$ is the solution to the homogeneous form of (2.7).

To use the lemma, we require the operator which takes A to $i\sum_{Z\in S_\Lambda(X)}[\tau_t^\Lambda(\Phi(Z)), A]$ to be norm preserving. This can be shown by observing that

$$i\sum_{Z\in S_\Lambda(X)}[\tau_t^\Lambda(\Phi(Z)), A] = i[\sum_{Z\in S_\Lambda(X)}\tau_t^\Lambda(\Phi(Z)), A].$$

Also, notice that since $\tau_t^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ is a $*$-algebra homomorphism and Φ is self-adjoint, $\tau_t^\Lambda(\Phi(Z))^* = \tau_t^\Lambda(\Phi(Z)^*) = \tau_t^\Lambda(\Phi(Z))$. Thus $\tau_t^\Lambda(\Phi(Z))$ is self-adjoint, which implies it is, itself, some interaction on Λ. We conclude that the sum of these interactions, $\sum_{Z\in S_\Lambda(X)}\tau_t^\Lambda(\Phi(Z))$, is also a Hamiltonian $H(t)$ on Λ. In A.2 we show in general that the operator $i[H(t), \cdot\,]$ defined by $x \mapsto i[H(t), x]$ is norm preserving, which implies that $[\tau_t^\Lambda(\Phi(Z)), f(t)]$ is norm-preserving. Therefore, we can apply Lemma 4 to the differential equation for $f'(t)$ given in (2.6).

Letting $t_0 = 0$, $f_0 = f(t_0) = [\tau_0^\Lambda(\tau_0^X(A)), B] = [A, B]$, we get the inequality

$$\|f(t) - T(t)f_0\| \leq \int_{\min\{0,t\}}^{\max\{0,t\}} \Big\|i\sum_{Z\in S_\Lambda(X)}\big[\tau_t^\Lambda(\tau_{-t}^X(A)), [\tau_t^\Lambda(\Phi(Z)), B]\big]\Big\| \mathrm{d}s,$$

where $T(t)f_0$ satisfies $(T(t)f_0)' = i\sum_{Z\in S_\Lambda(X)}[\tau_t^\Lambda(\Phi(Z)), T(t)f_0]$, the homogeneous form of (2.6). It is easy to see by the reverse triangle inequality that this implies

$$\|[\tau_t^\Lambda(\tau_{-t}^X(A)), B]\| \leq \|T(t)\cdot[A, B]\| + \int_{\min\{0,t\}}^{\max\{0,t\}} \sum_{Z\in S_\Lambda(X)} \|[\tau_s^\Lambda(\tau_{-s}^X(A)), [\tau_s^\Lambda(\Phi(Z)), B]]\| \mathrm{d}s.$$

Because $T(t)$ is norm invariant, the inequality above produces

$$\|[\tau_t^\Lambda(\tau_{-t}^X(A)), B]\| \leq \|[A, B]\| + 2\|A\| \sum_{Z\in S_\Lambda(X)} \int_{\min\{0,t\}}^{\max\{0,t\}} \|[\tau_s^\Lambda(\Phi(Z)), B]\| \mathrm{d}s. \tag{2.9}$$

Before moving on, it is worth noting the approximations made up to this point. So far, we have lost equality by moving the norm through the integral in the proof of Lemma 4, and by repeated triangle approximations in going from (2.6) to (2.9).

The strategy from here on will be to use equation (2.9), or a form similar to it, in an iterative manner to form an integral bound for expressions of the form $\|[\tau_s^\Lambda(K), B]\|$, where K is some observable supported on a subset of Λ. We begin by defining the norm of the commutator expression $\|[\tau_t^\Gamma(\cdot), B]\|$ relative to some support.

Definition 5. If Γ is some integer interval, with Hamiltonian H_Γ which generates Heisenberg dynamics given by τ_t^Γ, then given any observable B supported on Γ, we define the norm of the commutator $\|[\tau_t^\Gamma(\cdot), B]\|$ relative to some support $Z \subset \Gamma$ at time t, by

$$\sup_{\substack{A \in \mathcal{A}_Z \\ A \neq 0}} \frac{\|[\tau_t^\Gamma(A), B]\|}{\|A\|},$$

and denote this relative norm by $C_B^\Gamma(Z; t)$.

With this definition, we show the following proposition.

Proposition 5. *For Λ, X, B and Φ defined as before, the following relation holds*

$$C_B^\Lambda(X; t) \leq C_B^\Lambda(X; 0) + 2 \sum_{Z \in S_\Lambda(X)} \|\Phi(Z)\| \int_{\min\{0,t\}}^{\max\{0,t\}} C_B^\Lambda(Z; s) \mathrm{d}s,$$

and also for finite Z

$$C_B^\Lambda(Z; 0) \leq 2\|B\|\delta_Y(Z)$$

where δ_Y is defined by,

$$\delta_Y(Z) = \begin{cases} 1 & \text{if } Z \cap Y \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Proof. From (2.9), we know that for $A \neq 0$, dividing by $\|A\|$ gives

$$\frac{\|[\tau_t^\Lambda(\tau_{-t}^X(A)), B]\|}{\|A\|} \leq \frac{\|[A, B]\|}{\|A\|} + 2 \sum_{Z \in S_\Lambda(X)} \int_{\min\{0,t\}}^{\max\{0,t\}} \|[\tau_s^\Lambda(\Phi(Z)), B]\| \mathrm{d}s \tag{2.10}$$

By noting that $\tau_0^\Lambda(A) = e^{i0H_\Lambda} A e^{i0H_\Lambda} = A$, we get that $\frac{\|[A,B]\|}{\|A\|} = \frac{\|[\tau_0^\Lambda(A),B]\|}{\|A\|}$. Taking the supremum over non-zero $A \in \mathcal{A}_X$, we have $\frac{\|[A,B]\|}{\|A\|} \leq \sup_{A \in \mathcal{A}_X} \frac{\|[\tau_0^\Lambda(A),B]\|}{\|A\|} = C_B^\Lambda(X; 0)$. This permits us to replace $\frac{\|[A,B]\|}{\|A\|}$ in (2.10) by $C_B^\Lambda(X; 0)$, to get

$$\begin{aligned}
\frac{\|[\tau_t^\Lambda(\tau_{-t}^X(A)), B]\|}{\|A\|} &\leq C_B^\Lambda(X; 0) + 2 \sum_{Z \in S_\Lambda(X)} \int_{\min\{0,t\}}^{\max\{0,t\}} \|[\tau_s^\Lambda(\Phi(Z)), B]\| \mathrm{d}s \\
&= C_B^\Lambda(X; 0) + 2 \sum_{Z \in S_\Lambda(X)} \|\Phi(Z)\| \int_{\min\{0,t\}}^{\max\{0,t\}} \frac{\|[\tau_s^\Lambda(\Phi(Z)), B]\|}{\|\Phi(Z)\|} \mathrm{d}s, \quad \text{for } \Phi(Z) \neq 0 \\
&\leq C_B^\Lambda(X; 0) + 2 \sum_{Z \in S_\Lambda(X)} \|\Phi(Z)\| \int_{\min\{0,t\}}^{\max\{0,t\}} \sup_{\substack{A \in \mathcal{A}_Z \\ A \neq 0}} \frac{\|[\tau_s^\Lambda(A), B]\|}{\|A\|} \mathrm{d}s \quad \text{since } \Phi(Z) \in \mathcal{A}_Z \\
&\leq C_B^\Lambda(X; 0) + 2 \sum_{Z \in S_\Lambda(X)} \|\Phi(Z)\| \int_{\min\{0,t\}}^{\max\{0,t\}} C_B^\Lambda(Z; s) \mathrm{d}s.
\end{aligned} \tag{2.11}$$

Thus, the right-hand-side of (2.11) is an upper bound for $\frac{\|[\tau_t^\Lambda(\tau_{-t}^X(A)),B]\|}{\|A\|}$, $A \neq 0$. To show Proposition 5, we first note that $\|A\| = \|\tau_{-t}^X(A)\|$, for all $t \in \mathbb{R}$. So, we have

$$\frac{\|[\tau_t^\Lambda(\tau_{-t}^X(A)),B]\|}{\|\tau_{-t}^X(A)\|} \leq C_B^\Lambda(X;0) + 2\sum_{Z\in S_\Lambda(X)} \|\Phi(Z)\| \int_{\min\{0,t\}}^{\max\{0,t\}} C_B^\Lambda(Z;s)\mathrm{d}s$$

which implies

$$\sup_{\substack{\tau_{-t}^X(A)\in\mathcal{A}_X \\ \tau_{-t}^X(A)\neq 0}} \frac{\|[\tau_t^\Lambda(\tau_{-t}^X(A)),B]\|}{\|\tau_{-t}^X(A)\|} \leq C_B^\Lambda(X;0) + 2\sum_{Z\in S_\Lambda(X)} \|\Phi(Z)\| \int_{\min\{0,t\}}^{\max\{0,t\}} C_B^\Lambda(Z;s)\mathrm{d}s \tag{2.12}$$

We claim the supremum in (2.12) is equal to $C_B^\Lambda(X;t)$. To see this, notice $\tau_{-t}^X : \mathcal{A}_X \to \mathcal{A}_X$ is onto $\mathcal{A}_X$. Surjectivity holds because given any $A_0 \in \mathcal{A}_X$, there exists $C \in \mathcal{A}_X$, namely $C = \tau_t^X(A_0)$, such that $\tau_{-t}^X(C) = \tau_{-t}^X(\tau_t^X(A_0)) = A_0$. So $\tau_{-t}^X(\mathcal{A}_X) = \mathcal{A}_X$, which implies

$$\sup_{\substack{\tau_{-t}^X(A)\in\mathcal{A}_X \\ \tau_{-t}^X(A)\neq 0}} \frac{\|[\tau_t^\Lambda(\tau_{-t}^X(A)),B]\|}{\|\tau_{-t}^X(A)\|} = \sup_{\substack{A\in\mathcal{A}_X \\ A\neq 0}} \frac{\|[\tau_t^\Lambda(A),B]\|}{\|A\|} = C_B^\Lambda(X;t).$$

Subsequently, substituting back into (2.12) confirms the first part of Proposition 5.

Finally, to show $C_B^\Lambda(Z;0) \leq 2\|B\|\delta_Y(Z)$, notice for all $A \in \mathcal{A}_Z$ that

$$\frac{\|[\tau_0^\Lambda(A),B]\|}{\|A\|} = \frac{\|[A,B]\|}{\|A\|} \leq \begin{cases} 0 & \text{if } [A,B]=0 \\ 2\|B\| & \text{if } [A,B]\neq 0. \end{cases}$$

Because $B \in \mathcal{A}_Y$, if $Z \cap Y = \emptyset$, then $[A,B] = 0$. Thus $2\|B\|\delta_Y(Z)$ is an upper bound for $\frac{\|[\tau_0^\Lambda(A),B]\|}{\|A\|}$, which implies that $C_B^\Lambda(Z;0) \leq 2\|B\|\delta_Y(Z)$. □

With Proposition 5 proved, we can apply it to bound the expression $C_B^\Lambda(Z;s)$ by

$$C_B^\Lambda(Z;s) \leq 2\|B\|\delta_Y(Z) + 2\sum_{Z_1\in S_\Lambda(Z)} \|\Phi(Z_1)\| \int_{\min\{0,s\}}^{\max\{0,s\}} C_B^\Lambda(Z_1;s_1)\mathrm{d}s_1. \tag{2.13}$$

We can continue to repeatedly approximate the integrand term of (2.13) by applying Proposition 5, which leads to the series bound for $\|\tau_t^\Lambda(A),B]\|$ given below.

2.4 Series Bound for $\|[\tau^\Lambda(A),B]\|$

Proposition 6. *Given $\|[\tau^\Lambda(A),B]\|$ as defined in Theorem 3, if $N \geq 1$ then*

$$||[\tau_t^\Lambda(A),B]|| \leq 2||A||||B|| \left(\delta_Y(X) + \sum_{n=1}^N a_n(t)\right) + R_{N+1}(t) \tag{2.14}$$

where

$$a_n(t) = \frac{(2|t|)^n}{n!} \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_n\in S_\Lambda(Z_{n-1})} \delta_Y(Z_n) \prod_{j=1}^n ||\Phi(Z_j)|| \tag{2.15}$$

and

$$R_{N+1}(t) = \frac{(2|t|)^{N+1}}{(N+1)!}\|A\| \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_{N+1}\in S_\Lambda(Z_N)} \sup_{s\in[0,t]} ||[\tau_s^\Lambda(\Phi(Z_{N+1})), B]|| \times \prod_{j=1}^{N} ||\Phi(Z_j)||. \tag{2.16}$$

Further, the remainder term, $R_{N+1}(t) \to 0$ *as* $N \to \infty$.

A careful inductive proof of Proposition 6 is given in the appendix A.3, however, below we approximate $C_B^\Lambda(X;t)$ by iteratively replacing the integrand terms a couple times to demonstrate the mechanism. For notational ease, we will denote $S_\Lambda(Z_i)$ by S_i, for $i = 0, 1, \cdots$, where $Z_0 := X$. Also, we will take $t > 0$ so we can drop the min/max bounds on the integrals. Then, using (2.13) to bound $C_B^\Lambda(Z;s)$ from Proposition 5 gives

$$\begin{aligned}
&C_B^\Lambda(X;t)\\
&\quad\le C_B^\Lambda(X;0) + 2\sum_{Z_1\in S_0}\|\Phi(Z_1)\|\int_0^t C_B^\Lambda(Z_1;s)\mathrm{d}s_1 &(2.17)\\
&\quad\le 2\|B\|\delta_Y(X) +\\
&\qquad\quad 2\sum_{Z_1\in S_0}\|\Phi(Z_1)\|\int_0^t \left(2\|B\|\delta_Y(Z_1) + 2\sum_{Z_2\in S_1}\|\Phi(Z_2)\|\int_0^{s_1} C_B^\Lambda(Z_2,s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 &(2.18)\\
&\quad= 2\|B\|\left(\delta_Y(X) + 2\sum_{Z_1\in S_0}\|\Phi(Z_1)\|\delta_Y(Z_1)\int_0^t \mathrm{d}s_1\right) +\\
&\qquad\qquad 2^2\sum_{Z_1\in S_0}\|\Phi(Z_1)\|\sum_{Z_2\in S_1}\|\Phi(Z_2)\|\int_0^t\int_0^{s_1} C_B^\Lambda(Z_2;s_2)\mathrm{d}s_2\mathrm{d}s_1. &(2.19)
\end{aligned}$$

The term in parenthesis in (2.19) is $\delta_Y(X) + a_1(t)$ since the integral evaluates to t, and we notice that from the proof of Proposition 5

$$\begin{aligned}
C_B^\Lambda(Z_2;s_2) &= \sup_{\substack{K\in\mathcal{A}_{Z_2}\\ K\neq 0}} \frac{\|\tau_{s_2}^\Lambda(K), B]\|}{\|K\|} = \sup_{\substack{K\in\mathcal{A}_{Z_2}\\ K\neq 0}} \frac{\|\tau_{s_2}^\Lambda(\tau_{-s_2}^{Z_2}(K)), B]\|}{\|K\|}\\
&\le 2\|B\|\delta_Y(Z_2) + 2\sum_{Z_3\in S_2}\int_0^{s_2} \|[\tau_{s_3}^\Lambda(\Phi(Z_3)), B]\|\mathrm{d}s_3,
\end{aligned} \tag{2.20}$$

where the inequality holds by 2.9. Reinserting (2.20) into (2.19) gives

$$\begin{aligned}
C_B^\Lambda(X;t) \le\; & 2\|B\|(\delta_Y(X) + a_1(t)) + 2^2\sum_{Z_1\in S_0}\sum_{Z_2\in S_1}\|\Phi(Z_1)\|\|\Phi(Z_2)\|\times\\
& \int_0^t\int_0^{s_1}\left(2\|B\|\delta_Y(Z_2) + 2\sum_{Z_3\in S_2}\int_0^{s_2}\|[\tau_{s_3}^\Lambda(\Phi(Z_3)), B]\|\mathrm{d}s_3\right)\mathrm{d}s_2\mathrm{d}s_1.\\
=\; & 2\|B\|[\delta_Y(X) + a_1(t)] + 2\|B\|[\delta_Y(X) + a_2(t)] + R_3(t)/\|A\|\\
=\; & 2\|B\|[\delta_Y(X) + \sum_{n=1}^{2} a_n(t)] + R_3(t)/\|A\|.
\end{aligned} \tag{2.21}$$

Finally, since $\|[\tau_t^\Lambda(A), B]\|/\|A\| \le C_B^\Lambda(X;t)$, we have by multiplying equation (2.21) by $\|A\|$ that $\|[\tau_t^\Lambda(A), B]\|$ is bounded as in Proposition 6, with $N = 2$.

2.4.1 Bounding $a_n(t)$

Now that we have a obtained a series bound for $\|[\tau_t^\Lambda(A), B]\|$, we can use an $\mathfrak{F}$-function to bound the sequence terms $a_n(t)$ which will lead to a more tractable bound. Recall that given an $\mathfrak{F}$-function F, the F-norm of interaction is given by $\|\Phi\|_F = \sup_{x,y\in\mathbb{Z}} \sum_{Z\ni x,y} \frac{\|\Phi(Z)\|}{F(|x-y|)}$. So, for any $x, y \in \mathbb{Z}$, it is clear that $\|\Phi\|_F F(|x-y|) \ge \sum_{Z\ni x,y} \|\Phi(Z)\|$. Also recall the convolution property of F provides a way to bound $\sum_{z\in\mathbb{Z}} F(|x-z|)F(|z-y|)$ by $C_F F(|x-y|)$. These two properties allow the expression for $a_n(t)$ to be condensed.

Recall from Proposition 6, we have the definition

$$a_n(t) = \frac{(2|t|)^n}{n!} \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_n\in S_\Lambda(Z_{n-1})} \delta_Y(Z_n) \prod_{j=1}^n \|\Phi(Z_j)\|\,,$$

where $X, Y \subset \Lambda$ are taken to be disjoint. An *overcount* of $a_n(t)$ is then given by

$$a_n(t) \le \frac{(2|t|)^n}{n!} \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} \sum_{\substack{w_1\in\Lambda\\ w_2\in\Lambda\\ \vdots\\ w_{n-1}\in\Lambda}} \sum_{\substack{Z_1\subset\Lambda:\\ x\in Z_1\\ w_1\in Z_1}} \sum_{\substack{Z_2\subset\Lambda:\\ w_1\in Z_2\\ w_2\in Z_2}} \cdots \sum_{\substack{Z_n\subset\Lambda:\\ w_{n-1}\in Z_n\\ y\in Z_n}} \prod_{j=1}^n \|\Phi(Z_j)\|\,, \tag{2.22}$$

since the elements in each intersection $X\cap Z_1\,, Z_1\cap Z_2\,, \ldots, Z_n\cap Y$ are summed over at least $|\partial_\Phi(X)|$ times. Then, by the properties of F we obtain

$$\begin{aligned}
a_n(t) &\le \frac{(2|t|)^n}{n!} \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} \sum_{\substack{w_1\in\Lambda\\ w_2\in\Lambda\\ \vdots\\ w_{n-1}\in\Lambda}} \sum_{\substack{Z_1\subset\Lambda:\\ x\in Z_1\\ w_1\in Z_1}} \|\Phi(Z_1)\| \sum_{\substack{Z_2\subset\Lambda:\\ w_1\in Z_2\\ w_2\in Z_2}} \|\Phi(Z_2)\| \cdots \sum_{\substack{Z_n\subset\Lambda:\\ w_{n-1}\in Z_n\\ y\in Z_n}} \|\Phi(Z_n)\| \\
&\le \frac{(2|t|)^n}{n!} \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} \sum_{\substack{w_1\in\Lambda\\ w_2\in\Lambda\\ \vdots\\ w_{n-1}\in\Lambda}} \Big(\|\Phi\|_F F(|x-w_1|)\Big)\Big(\|\Phi\|_F F(|w_1-w_2|)\Big)\cdots\Big(\|\Phi\|_F F(|w_{n-1}-y|)\Big) \\
&= \frac{(2\|\Phi\|_F\,|t|)^n}{n!} \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} \sum_{w_{n-1}\in\Lambda} \sum_{w_{n-2}\in\Lambda} \cdots \\
&\qquad\qquad \sum_{w_1\in\Lambda} F(|x-w_1|)F(|w_1-w_2|)F(|w_2-w_3|)\cdots F(|w_{n-1}-y|) \\
&\le \frac{(2\|\Phi\|_F\,|t|)^n}{n!} \cdot C_F \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} \sum_{w_{n-1}\in\Lambda} \sum_{w_{n-2}\in\Lambda} \cdots \sum_{w_2\in\Lambda} F(|x-w_2|)F(|w_2-w_3|)\cdots F(|w_{n-1}-y|) \\
&\;\vdots \\
&\le \frac{(2\|\Phi\|_F\,|t|)^n}{n!} \cdot {C_F}^{n-1} \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} F(|x-y|)
\end{aligned}$$

$$= \frac{(2\|\Phi\|_F\, C_F\, |t|)^n}{n!\, C_F} \sum_{\substack{x\in\partial_\Phi(X)\\ y\in Y}} F(|x-y|)\,, \tag{2.23}$$

where the vertical dots denote the repeated application of the convolution property. Thus, (2.22) can finally be written as

$$a_n(t) \leq \frac{(v|t|)^n}{n!\, C_F} \sum_{x\in\partial_\Phi(X)} \sum_{y\in Y} F(|x-y|) \tag{2.24}$$

where

$$v = 2\|\Phi\|_F\, C_F.$$

Given this bound on the $a_n(t)$ terms, it follows that

$$\sum_{n=1}^{\infty} a_n(t) \leq \sum_{n=1}^{\infty} \frac{(v|t|)^n}{n! C_F} \sum_{x\in\partial_\Phi(X)} \sum_{y\in Y} F(|x-y|) = (e^{v|t|}-1) \sum_{x\in\partial_\Phi(X)} \sum_{y\in Y} F(|x-y|).$$

A similar argument given in Appendix A.3.1 shows that $R_N \leq 2\|A\|\|B\| \frac{(v|t|)^N}{N! C_F} |\partial_\Phi(X)| \|F\|$ which implies $\lim_{N\to\infty} R_N(t) = 0$.

Now $\delta_Y(X) = 0$ because $X \cap Y = \emptyset$ by assumption, so we immediately obtain from Proposition 6 the estimate

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F}(e^{v|t|}-1) \sum_{x\in\partial_\Phi(X)} \sum_{y\in Y} F(|x-y|). \tag{2.25}$$

We can also consider moving the dynamics from A onto B to get the bound

$$\|[\tau_{-t}^\Lambda(B), A]\| \leq \frac{2\|A\|\|B\|}{C_F}(e^{v|t|}-1) \sum_{y\in\partial_\Phi(Y)} \sum_{x\in X} F(|x-y|). \tag{2.26}$$

Since $\|[\tau_t^\Lambda(A), B]\| = \| - [e^{-itH_\Lambda}\tau_t^\Lambda(A)e^{itH_\Lambda}, e^{-itH_\Lambda}Be^{itH_\Lambda}]\| = \|[\tau_{-t}^\Lambda(B), A]\|$, we know that $\|[\tau_t^\Lambda(A), B]\|$ is bounded by both inequalities (2.25) and (2.26), so taking a minimum upper bound immediately confirms Theorem 3.

It should be noted that the bound from Theorem 3 is only relevant in time regimes where $\frac{e^{v|t|}-1}{C_F} D(X,Y) < 1$, since outside this region the trivial bound $2\|A\|\|B\|$ will provide better estimates for $\|[\tau_t^\Lambda(A), B]\|$. Also, as noted in the introduction to $\mathfrak{F}$-functions, we can use a different $\mathfrak{F}$-function given by $F_\alpha(r) = e^{-\alpha r}F(r),\ \alpha \geq 0$. In this case, we will have

$$D(X,Y) \leq \min\{|\partial_\Phi(X)|, |\partial_\Phi(Y)|\}\|F_0\|e^{-\alpha d(X,Y)},$$

where $d(X,Y)$ is the distance between the sets X and Y in the standard sense. Using this idea, we get a better notion of the quantity $v = 2\|\Phi\|_{F_\alpha} C_{F_\alpha}$ as a velocity, since

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|B\|}{C_{F_\alpha}} \min\{|\partial_\Phi(X)|, |\partial_\Phi(Y)|\}\|F_0\|e^{v|t|-\alpha d(X,Y)}, \tag{2.27}$$

which implies the localizing bound (2.27) decays exponentially in separation of X and Y with rate $\alpha > 0$, and the bound grows at rate v with time.

2.5 Short Range Interactions

The Lieb-Robinson bound given in Theorem 3 makes no assumptions on the behavior or strength of the interaction Φ, save that it has bounded $\mathfrak{F}$-norm. By restricting the distance between lattice points on which an interaction will act non-trivially, such as in the case of nearest-neighbor interactions or generally any short-range interaction, we expect time-evolved observables to remain localized for longer periods of time. Indeed, this is what Theorem 7, given below, implies.

Theorem 7 (Lieb-Robinson Bound for Short-Range Interactions)**.** *Let $X, Y \subset \Lambda = [a, b]$ be finite and disjoint integer intervals and let F be an $\mathfrak{F}$-function with convolution constant C_F. Suppose Φ in $\mathcal{B}_F(\Lambda)$ is a short range interaction on $\mathcal{P}(\Lambda)$ with range $d = \max_{Z\in\Lambda}\{\max_{x,y\in Z}|x-y| : \Phi(Z) \neq 0\}$. Finally, denote the time evolution generated by the Heisenberg dynamics with Hamiltonian*

$$H_\Lambda = \sum_{Z\subset\Lambda} \Phi(Z)$$

by $\tau_t^\Lambda(\cdot)$ and define the distance between the sets X, Y in range-units by the expression

$$\Delta(X,Y) := \left\lceil \frac{1}{d} \min_{\substack{x\in X\\ y\in Y}} \Big\{ |x-y| \Big\} \right\rceil .$$

If $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ are any two local observables, then we may estimate

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F} D(X,Y) \Big(e^{2\|\Phi\|_F\, C_F\, |t|} - P_{\Delta(X,Y)}(t) \Big) ,$$

where

$$D(X,Y) := \min\Bigg\{ \sum_{x\in\partial_\Phi(X)} \sum_{y\in Y} F(|x-y|) \ , \ \sum_{y\in\partial_\Phi(Y)} \sum_{x\in X} F(|x-y|) \Bigg\}$$

and

$$P_{\Delta(X,Y)}(t) := \sum_{n=0}^{\Delta(X,Y)-1} \frac{(2\|\Phi\|_F\, C_F\, |t|)^n}{n!} .$$

Before proving the theorem, it is useful to make a few observations. For the nearest-neighbor interaction where $\Phi(Z) = 0$ unless $Z = \{x, x+1\} \subset \Lambda$, the interaction range given by d is simply 1. Also, since X and Y are integer intervals, the Φ-boundaries of X and Y are simply the interval endpoints, so each has cardinality at most 2. Using $\mathfrak{F}$-function F_α, the nearest neighbor interaction can be estimated by

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{4\|A\|\|B\|\|F_0\|}{C_{F_\alpha}} \Bigg(e^{v|t|} - \sum_{n=0}^{d(X,Y)-1} \frac{(v|t|)^n}{n!} \Bigg) e^{-\alpha d(X,Y)},$$

where v is the Lieb-Robinson bound given by $2\|\Phi\|_{F_\alpha} C_{F_\alpha}$. Notice in this case if $v|t| < 1$, the bound is substantially reduced compared to the standard Lieb-Robinson bound because $e^{v|t|} - \sum_{n=0}^{d(X,Y)-1}$ is of order $\mathcal{O}((v|t|)^{d(X,Y)})$, so the truncation removes the majority of the contributing terms from the exponential.

Proof of Theorem 7. To prove Theorem 7, we refer to Proposition 6, which provides a series bound for $\|[\tau_t^\Lambda(A), B]\|$ expressed by $\sum_{n=1}^\infty a_n(t)$, where

$$a_n(t) = \frac{(2|t|)^n}{n!} \sum_{Z_1 \in S_\Lambda(X)} \sum_{Z_2 \in S_\Lambda(Z_1)} \cdots \sum_{Z_n \in S_\Lambda(Z_{n-1})} \delta_Y(Z_n) \prod_{j=1}^n \|\Phi(Z_j)\|.$$

The $\delta_Y(Z_n)$ term implies that $Z_n \cap Y \neq \emptyset$ is a necessary condition for non-vanishing $a_n(t)$. Further, the factor $Z_1 \in S_\Lambda(X)$ implies $Z_1 \cap X$, while $Z_2 \in S_\Lambda(Z_1), \ldots, Z_N \in S_\Lambda(Z_{n-1})$ requires that each Z_i intersects non-trivially with Z_{i+1} for $i = 1, \ldots, n-1$. Because the range of Φ is d, it follows that the width W_i of each Z_i, in the sense that $W_i = \max_{x,y}\{|x-y| : x, y \in Z_i\}$, must be less than d to satisfy $\|\Phi(Z_i)\| \neq 0$. Thereby, the index n must equal or exceed the value $d(X,Y)/d$ before $a_n(t) \neq 0$. Hence, we conclude that $\sum_{n=1}^\infty a_n(t) = \sum_{\Delta(X,Y)}^\infty a_n(t)$, from which Theorem 7 follows immediately. □

3 Coupled Interval Bounds via the Interaction Picture

In section 2, we primarily develop bounds on $\|[\tau_t^\Lambda(A), B]\|$ without assuming any additional knowledge of the behavior on the interval Λ. However, in many spin systems it is realistic to know that Λ is composed of subintervals with well understood (but potentially varying) behavior, each coupled together by some interaction. This motivates investigating coupled systems by considering each component interval. One way to do this is to decompose the dynamics into two parts: the free dynamics who's behavior is known, and the interaction dynamics which is given by the coupling interactions. We will use the interaction picture discussed in section 1.4 to develop Lieb-Robinson like bounds on coupled systems.

3.1 Two-Interval Case

Consider two consecutive disjoint intervals $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$. We identify the right endpoint of Λ_1 by a and the left endpoint of Λ_2 by b. Further, suppose that there are quantum spin systems defined on the intervals Λ_i with Hamiltonians $H_i \in \mathcal{A}_{\Lambda_i}$ generated by nearest neighbor interactions Φ_i. Now, suppose Ψ is a nearest neighbor coupling of the intervals, that is $\Psi \in \mathcal{A}_{\{a,b\}}$. Then, on each system we can assume that we have a Lieb-Robinson bound of the following form:

$$\|[\tau_t^{\Lambda_i}(A_i), B_i]\| \leq 2C\|A_i\|\|B_i\| \min\{|\partial_{\Phi_i} X_i|, |\partial_{\Phi_i} Y_i|\}(e^{v_i|t|} - 1)e^{-\mu d(X_i, Y_i)},$$

where $X_i = \text{supp}(A_i)$ and $Y_i = \text{supp}(B_i)$ are disjoint subsets of Λ_i, and C is independent of i, A_i and B_i.

We hope to form a Lieb-Robinson like bound on the dynamics of the interval $\Lambda = \Lambda_1 \cup \Lambda_2$ by using the interaction picture in the following form. If $H = H_1 + H_2 + \Psi$ is the Hamiltonian, we will decompose the Heisenberg dynamics τ_t corresponding to H as $\tau_t^0 \circ \tau_t^I$, where τ_t^0 is the Heisenberg dynamics generated by the free Hamiltonian $H_0 = H_1 + H_2$, and the interaction dynamics are determined by $\tau_t^I(A) = U^*(t)AU(t)$, where $U(t)$ satisfies

$$\frac{dU}{dt} = -iU(t)\tau_{-t}^0(\Psi) \equiv -iU(t)\Psi(-t), \quad U(0) = \mathbb{1}.$$

We will use the fact that,

$$\frac{d}{dt}\tau_t^I(A) = i\Psi(-t)U^*(t)AU(t) - iU^*(t)AU(t)\Psi(-t) = i[\Psi(-t), \tau_t^I(A)]$$

to prove the following lemma.

Lemma 8. *Let A and B be local observables. Then for all $t \in \mathbb{R}$, we may estimate*

$$\|[\tau_t^I(A), B]\| \leq \|[A, B]\| + 2\|A\| \int_{\min(0,t)}^{\max(0,t)} \|[B, \Psi(-s)]\| \mathrm{d}s$$

Proof. We first notice by the Jacobi Identity that,

$$\begin{aligned}\frac{d}{dt}[\tau_t^I(A), B] &= i[[\Psi(-t), \tau_t^I(A)], B] \\ &= i[\Psi(-t), [\tau_t^I(A), B]] - i[[B, \Psi(-t)], \tau_t^I(A)].\end{aligned} \tag{3.1}$$

By the proof given in A.2, the first term in (3.1) is norm preserving, so we have by Lemma 4 that

$$\begin{aligned}\|[\tau_t^I(A), B]\| &\leq \|[A, B\| + \int_{\min(0,t)}^{\max(0,t)} \|[[B, \Psi(-s)], \tau_s^I(A)]\| \mathrm{d}s \\ &\leq \|[A, B]\| + 2\|A\| \int_{\min(0,t)}^{\max(0,t)} \|[B, \Psi(-s)]\| \mathrm{d}s.\end{aligned}$$

□

Proposition 9. *Let A and B be local observables such that $X = supp(A) \subseteq \Lambda_1 \setminus \{a\}$ and $Y = supp(B) \subseteq \Lambda_2 \setminus \{b\}$. Then for all $t \in \mathbb{R}$ we have,*

$$\|[\tau_t(A), B]\| \leq 4C\|A\|\|B\|\|\Psi\||t| \min\{(e^{v_1|t|} - 1)e^{-\mu d(X,a)}, (e^{v_2|t|} - 1)e^{-\mu d(b,Y)}\}$$

Proof. Lemma 8 implies that we can bound the interaction dynamics, motivating us to move the free dynamics onto B. Since $\tau_{-t}^0(B)$ is entirely localized in $\mathcal{A}_{\Lambda_2}$, we can use Lemma 8 to say

$$\begin{aligned}\|[\tau_t(A), B]\| = \|[\tau_t^I(A), \tau_{-t}^0(B)]\| \leq \|[A, \tau_{-t}^0(B)]\| + 2\|A\| &\int_{\min(0,t)}^{\max(0,t)} \|[\tau_{-t}^0(B), \Psi(-s)]\| \mathrm{d}s \\ = 2\|A\| &\int_{\min(0,t)}^{\max(0,t)} \|[\tau_{s-t}^0(B), \Psi]\| \mathrm{d}s \\ \leq 4C\|A\|\|B\|\|\Psi\| e^{-\mu d(b,Y)} &\int_{\min(0,t)}^{\max(0,t)} (e^{v_2|t-s|} - 1) \mathrm{d}s \quad (3.2) \\ = 4C\|A\|B\|\|\Psi\| &\frac{1}{v_2}(e^{v_2|t|} - v_2|t| - 1)e^{-\mu d(b,Y)} \quad (3.3) \\ \leq 4C\|A\|\|B\|\|\Psi\||t|&(e^{v_2|t|} - 1)e^{-\mu d(b,Y)},\end{aligned}$$

where (3.2) holds since $\|[\tau_{s-t}^0(B), \Psi]\| \leq 2C\|B\|\|\Psi\|(e^{v_2|t-2|} - 1)e^{-\mu d(b,Y)}$ by assumption of a Lieb-Robinson bound on interval Λ_2.

By a symmetric argument it follows that

$$\|[\tau_t(A), B]\| \leq 4C\|A\|\|B\|\|\Psi\||t|(e^{v_1|t|} - 1)e^{-\mu d(X,a)}$$

from which Lemma 9 follows immediately. □

Before moving on, it is worth noting that the approximation after (3.3) is used here to make further computation reasonable. However, as we will discuss in section 3.4, keeping the bound in the form of (3.2) or (3.3) is desirable.

3.2 Disordered Case

Now suppose that we have three disjoint consecutive intervals $\Lambda_1, \Lambda_2, \Lambda_3 \subset \mathbb{Z}$, with Hamiltonians $H_i \in \mathcal{A}_{\Lambda_i}$ and nearest neighbor couplings Ψ_1 (coupling Λ_1 to Λ_2) and Ψ_2 (coupling Λ_2 to Λ_3). We will assume that the quantum spin system on Λ_2 exhibits localization in the form of a zero-velocity Lieb-Robinson bound [2], which for convenience we assume holds deterministically:

$$\|[\tau_t^{\Lambda_2}(A), B]\| \leq 2C\|A\|\|B\|e^{-\mu d(X,Y)}$$

for all $t \in \mathbb{R}$ and any local observables with $X = \mathrm{supp}(A), Y = \mathrm{supp}(B)$, with C and μ independent of A, B. With this convention, we pose the following lemma.

Lemma 10. *Let $\Lambda = \Lambda_1 \cup \Lambda_2$. Then for observables $A \in \mathcal{A}_{\Lambda_1}$ and $B \in \mathcal{A}_{\Lambda_2}$ and all $t \in \mathbb{R}$,*

$$\|[\tau_t^{\Lambda}(A), B]\| \leq 4C\|A\|\|B\|\|\Psi_1\||t|e^{-\mu d(supp(B), supp(\Psi_1))}$$

Proof. Use the interaction picture $\tau_t^\Lambda = \tau_t^0 \circ \tau_t^I$, where $H_0 = H_1 + H_2$ and $H_I = \Psi_1$. By Lemma 8,

$$\begin{aligned}\|[\tau_t^{\Lambda}(A), B]\| = \|[\tau_t^I(A), \tau_{-t}^0(B)] &\leq \|[A, \tau_{-t}^0(B)]\| + 2\|A\| \int_{\min(0,t)}^{\max(0,t)} \|[\tau_{-t}^0(B), \Psi_1(-s)]\| \mathrm{d}s \\ &= 2\|A\| \int_{\min(0,t)}^{\max(0,t)} \|[\tau_{s-t}^0(B), \Psi_1]\| \mathrm{d}s \\ &= 4C\|A\|\|B\|\|\Psi_1\||t|e^{-\mu d(\mathrm{supp}(B), \mathrm{supp}(\Psi_1))}\end{aligned}$$

□

We can now look at a three interval case, where Λ_2 separates Λ_1 from Λ_3. Given that the dynamics on Λ_2 are subject to a zero Lieb-Robinson velocity, we expect observables supported on Λ_1 and Λ_3 to be isolated from each other. Indeed, Proposition 11 below implies that a Lieb-Robinson type bound on this system grows quadratically in time as opposed to exponentially.

Proposition 11. *Suppose that $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_3$, with $\tau_t = \tau_t^\Lambda$ where $\Lambda = \cup_{i=1}^3 \Lambda_i$. Then,*

$$\|[\tau_t(A), B]\| \leq 4C\|A\|\|B\|\|\Psi_1\|\|\Psi_2\|t^2 e^{-\mu|\Lambda_2|}$$

Proof. Let $\tilde{\Lambda} = \Lambda_1 \cup \Lambda_2$. We use the interaction picture with $H_0 = H_1 + \Psi_1 + H_2 + H_3$ and $H_I = \Psi_2$. By Lemma 8,

$$\begin{aligned}\|[\tau_t(A), B]\| = \|[\tau_{-t}(B), A]\| = \|[\tau_{-t}^I(B), \tau_t^0(A)]\| \\ &\leq \|[B, \tau_t^0(A)]\| + 2\|B\| \int_{\min(0,-t)}^{\max(0,-t)} \|[\tau_t^0(A), \Psi_2(-s)]\| \mathrm{d}s \\ &= 2\|B\| \int_{\min(0,-t)}^{\max(0,-t)} \|[\tau_{s+t}^0(A), \Psi_2]\| \mathrm{d}s \\ &= 2\|B\| \int_{\min(0,-t)}^{\max(0,-t)} \|[\tau_{s+t}^{\tilde{\Lambda}}(A), \Psi_2]\| \mathrm{d}s \\ &\leq 8C\|A\|\|B\|\|\Psi_1\|\|\Psi_2\|e^{-\mu d(\mathrm{supp}(\Psi_1), \mathrm{supp}(\Psi_2))} \int_{\min(0,-t)}^{\max(0,-t)} |s+t| \mathrm{d}s \\ &= 4C\|A\|\|B\|\|\Psi_1\|\|\Psi_2\|t^2 e^{-\mu|\Lambda_2|},\end{aligned}$$

where we used Proposition 10 on the second to last line. □

3.3 Coupled Identical Chains

In this section, we study the effect dissimilar coupling strengths have on localizing the dynamics on spin chains. Here, we assume that we have a system of N integer intervals, $\Lambda_i = [p_i, q_i]$, each of length n and that the Hamiltonians $H_i \in \mathcal{A}_{\Lambda_i}$ determine the dynamics on each subinterval. For simplicity, we assume that there is a common $\mathfrak{F}$-function F_μ and Lieb-Robinson velocity v on each interval generated by a nearest-neighbor interaction. Each system is coupled together by a nearest-neighbor coupling Ψ_i (where Ψ_i joins interval Λ_i to Λ_{i+1}), and we assume the couplings are either very weak or very strong relative to the interval velocity. Notice if the couplings are entirely weak, meaning $\|\Psi\| := \max\{\|\Psi_i\|\} \ll v$, the Lieb-Robinson velocity of the entire coupled system is simply v while on the other hand if there exists strong couplings, then $\|\Psi\| := \max\{\|\Psi_i\|\} \gg v$ and the Lieb-Robinson velocity of the entire system is $2\|\Psi\|_{F_\mu} C_{F_\mu} := V \gg v$. In both these cases, we are interested in seeing whether the results from the previous propositions will produce a better bound than the standard one.

Proposition 12. *Suppose that $A \in \mathcal{A}_{\{p_1\}}$ and $B \in \mathcal{A}_{\{q_N\}}$. If $N \geq 2$, we may estimate*

$$\|[\tau_t(A), B]\| \leq \frac{2\|A\|\|B\|\|F_0\|}{C_{F_\mu}} \left(\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v} \right) \left(e^{v|t|} - P_{N-1}(v|t|) \right) e^{-\mu n}, \tag{3.4}$$

where $P_{N-1}(v|t|) = \sum_{j=0}^{N-1} \frac{(v|t|)^j}{j!}$.

Before proving Proposition 12, we notice that the bound is similar to the estimate for short-range interactions given in Theorem 7. The primary difference is that here, the truncated polynomial is of degree $N-1$, where N is the number of subintervals, while in Theorem 7 the truncated polynomial is of degree $\Delta(X,Y)-1$, where $\Delta(X,Y)$ is the distance between the supports of A and B in range-units. Thus, in some way each subinterval in the coupled system acts as a single interaction of range n.

Proof. The proof follows by induction. The base case (N=2) follows by Proposition 9, since letting $C = \|F_0\|/C_{F_\mu}$ we have by equation (3.3)

$$\begin{aligned} \|[\tau_t(A), B]\| &\leq 4C\|A\|\|B\|\|\Psi_1\| \frac{1}{v}(e^{v|t|} - v|t| - 1) \min\{e^{-\mu d(X,a)}, e^{-\mu d(b,Y)}\} \\ &= 2C\|A\|\|B\| \left(\frac{2\|\Psi_1\|}{v} \right) (e^{v|t|} - v|t| - 1) e^{-\mu n}. \end{aligned}$$

Now suppose the result holds for $N \geq 2$ intervals and that we consider the interval $\Lambda = \cup_{i=1}^{N} \Lambda_i$ as a single system. If we couple Λ to Λ_{N+1} with Ψ_N and regard $H_0 = \sum_{i=1}^{N+1} H_i + \sum_{i=1}^{N-1} \Psi_i$ and $H_I = \Psi_N$ as the Hamiltonians which generate the free and interaction dynamics τ^0 and τ^I, respectively, then by following the proof of Proposition 9 we see

$$\begin{aligned}
\|[\tau_t(A), B]\| &\leq 2\|B\| \int_{\min(0,t)}^{\max(0,t)} \|[\tau^0_{-t}(A), \Psi_N(-s)]\| \mathrm{d}s \\
&= 2\|B\| \int_{\min(0,t)}^{\max(0,t)} \|[\tau_{s-t}(A), \Psi_N]\| \mathrm{d}s \\
&\leq 2\|B\| 2C\|A\|\|\Phi_N\| \left(\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v} \right) e^{-\mu n} \int_{\min(0,t)}^{\max(0,t)} \left(e^{v|t-s|} - \sum_{n=0}^{N-1} \frac{(v|t-s|)^n}{n!} \right) \mathrm{d}s \\
&= 2\|B\| 2C\|A\|\|\Phi_N\| \left(\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v} \right) e^{-\mu n} \int_{\min(0,t)}^{\max(0,t)} \left(\sum_{j=N}^{\infty} \frac{(v|t-s|)^j}{j!} \right) \mathrm{d}s \\
&= 2C\|A\|\|B\| \left(\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v} \right) 2\|\Psi_N\| \frac{1}{v} \sum_{j=N}^{\infty} \frac{(v|t|)^{j+1}}{(j+1)!} e^{-\mu n} \\
&= 2C\|A\|\|B\| \left(\prod_{i=1}^{N} \frac{2\|\Psi_i\|}{v} \right) \left(e^{v|t|} - \sum_{j=0}^{N} \frac{(v|t|)^j}{j!} \right) e^{-\mu n},
\end{aligned}$$

which confirms the inductive step. Thus, Proposition 12 holds for all $N \geq 2$. □

To see when (3.4) is better than the standard Lieb-Robinson bound, we first suppose a weak coupling. In this case it is sufficient to require

$$\left(\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v} \right) \left(e^{v|t|} - \sum_{j=0}^{N-1} \frac{(v|t|)^j}{j!} \right) e^{-\mu n} \leq (e^{v|t|} - 1) e^{-\mu N n}.$$

In this form, it is difficult to analyze which t satisfy the inequality. However, if we recall that in this system, the maximum range for any interaction is n, we may instead compare when (3.4) is better than the short-range interaction Lieb-Robinson bound given in Theorem 7. This will hold whenever

$$\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v} \leq e^{-\mu n(N-1)}$$

and is certainly satisfied if $\|\Psi\| \leq \frac{ve^{-\mu n}}{2}$. Notice that this condition is independent of t, so it holds deterministically! It is worth noting that if $\|\Psi_i\| \ll \|\Psi\|$ for many i, the interval width n is permitted to be large, so this bound is not limited to systems of short range interactions. Such a situation is described in section 3.4. To see how much of an improvement this is over the standard Lieb-Robinson bound, we present plots in Figures 1 and 2 comparing the standard, short-range, and weakly coupled Lieb-Robinson bounds.

Note that we plot the bounds with the factor $\frac{2\|A\|\|B\|\|F_0\|}{C_{F_\mu}}$ removed for ease of comparison. That is, we are comparing $(e^{v|t|} - 1)e^{-\mu n N}$ plotted in green, $(e^{v|t|} - P_{N-1}(v|t|))e^{-\mu n N}$ in red, and

$$\left(\prod_{i=1}^{N-1} \frac{2\|\Psi\|}{v} \right) \left(e^{v|t|} - P_{N-1}(v|t|) \right) e^{-\mu n}$$

plotted in blue. Recall that here, N represents the number of intervals, n is the individual interval width, v is the Lieb-Robinson velocity on each interval, μ is the $\mathfrak{F}$-function constant, and $\|\Psi\|$ is the maximum coupling interaction size.

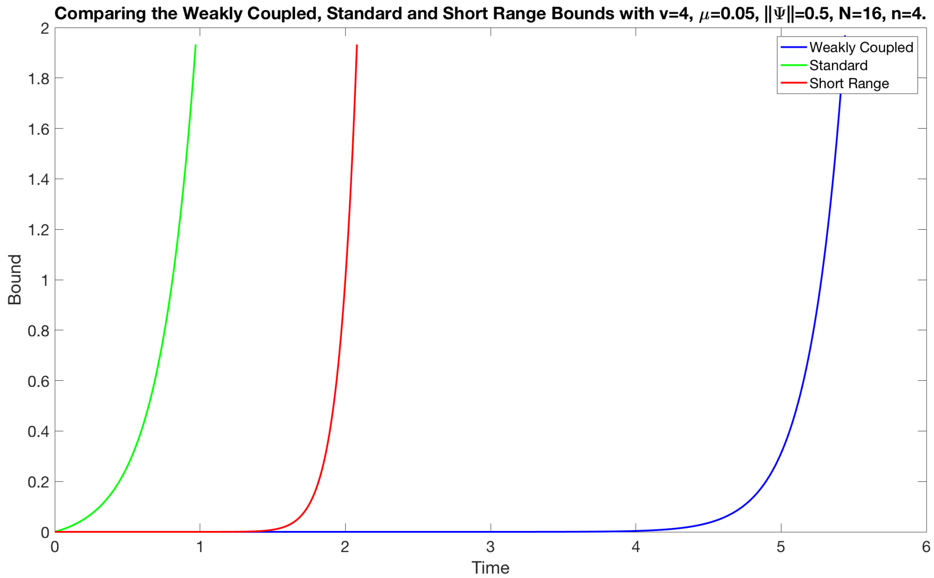

Figure 1: Comparing the standard and short-range Lieb-Robinson (LR) bounds with the estimate provided in Proposition 12 for a weakly coupled system of $N = 16$ identical intervals of width $n = 4$. Here, the interval LR-velocity is $v = 4$, $\mu = 0.05$, and each coupling is taken to be the maximum interaction strength $\|\Psi\| = 0.5$.

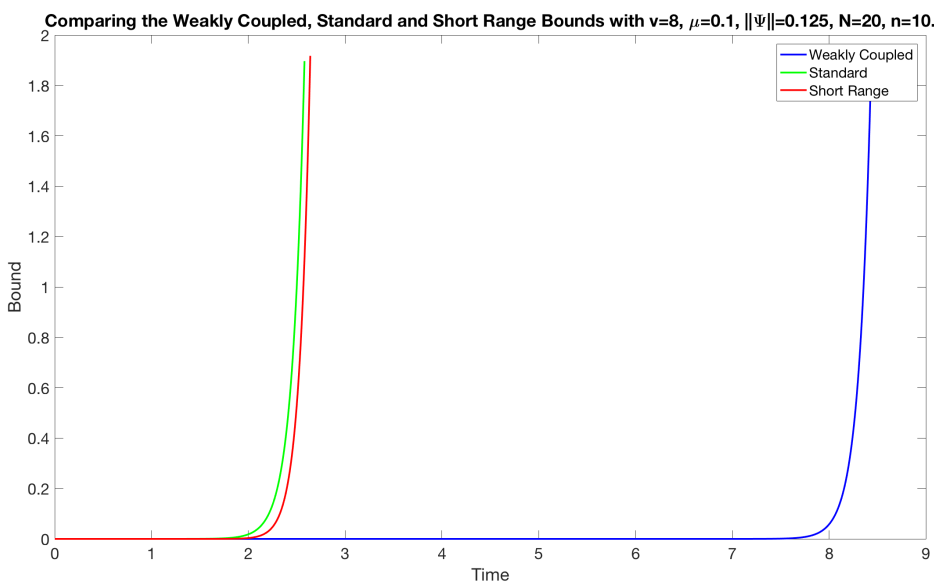

Figure 2: Comparing the standard and short-range Lieb-Robinson (LR) bounds with the estimate provided in Proposition 12 for a weakly coupled system of $N = 20$ identical intervals of width $n = 10$. Here, the interval LR-velocity is $v = 8$, maximum interaction strength $\|\Psi\| = 0.125$, and $\mu = 0.1$.

Figures 1 and 2 clearly show that for weakly coupled systems, the bound given in Proposition

12 provides a significantly better estimate for $\|[\tau_t(A), B]\|$ than both the standard Lieb-Robinson and short-range bounds. Moreover, in these estimates we have neglected the variability of the size of the individual Ψ_i. Accounting for this should produce even tighter bounds, as is shown in the next section.

3.4 Sparsely Dispersed Weak Couplings

In this section, we are interested in the effect weak couplings have on localizing the dynamics on spin chains. We consider a system of intervals coupled together by increasingly weaker couplings. Let the system begin with a single interval Λ of width $|\Lambda| = n = 2^p$, where p is a large integer. On the first step, we divide the interval into two disjoint pieces Λ_1 and Λ_2, coupled by an interaction Ψ_1 of strength $\|\Psi_1\| = 1/2$. On the second step, we divide each of Λ_1 and Λ_2 into two intervals, and coupled them with Ψ_1 while replacing the original Ψ_1 by Ψ_2 of strength $\|\Psi_2\| = 1/4$. By letting the strength $\|\Psi_i\| = 1/2^i$, we can say in general that at the k-th iteration (for $k \ll p$), our system will have 2^k intervals of width $n/2^k$, coupled together by 2^{k-1} interactions of type Ψ_1, 2^{k-2} interactions of type Ψ_2, $\cdots$, 2^{k-i} interactions of type Ψ_i, $\cdots$, 2^{k-k} interactions of type Ψ_k, as is shown in Figure 3.

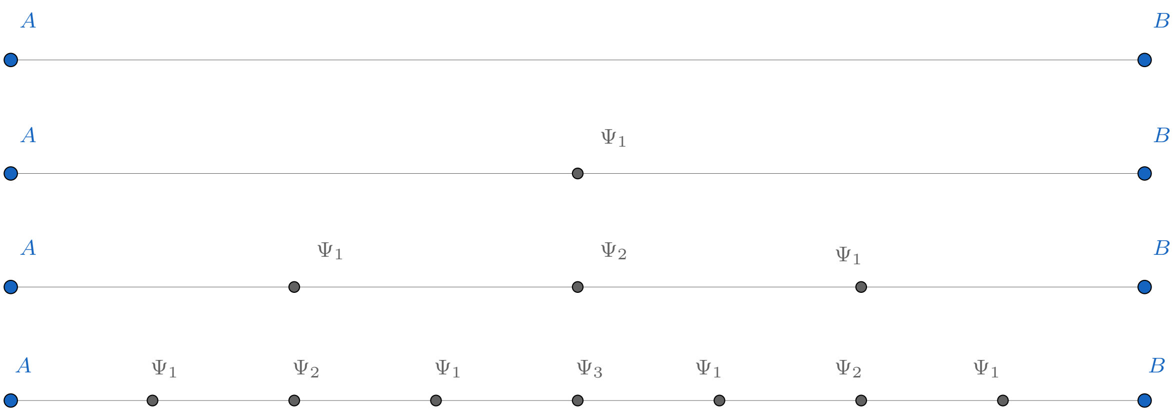

Figure 3: Distribution of couplings for $k = 0, 1, 2, 3$.

Remark 1. Note that instead of dividing the intervals on each step, we can form a similar system by iteratively joining two chains from step $k-1$ with a coupling Ψ_k to form the k-th chain. This creates an identical system, except that the distance between couplings will remain constant.

We can study the product in inequality (3.4) from Proposition 12 for this new system. Since at step k we have 2^k intervals and $2^k - 1$ couplings, we are considering the product

$$\begin{aligned}
&(2/v)^{2^k-1}\|\Psi_1\|^{2^{k-1}}\|\Psi_2\|^{2^{k-2}}\cdots|\Psi_k\| \\
=&(2/v)^{2^k-1}(1/2)^{2^{k-1}}(1/2^2)^{2^{k-2}}\cdots 1/2^k \\
=&(2/v)^{2^k-1}\prod_{i=1}^{k}\left(\frac{1}{2^i}\right)^{2^{k-i}} \\
=&(2/v)^{2^k-1}2^{-\sum_{i=1}^{k} i2^{k-i}}
\end{aligned} \tag{3.5}$$

We can evaluate the sum in (3.5) as

$$\sum_{i=1}^{k} i2^{k-i} = 2^{k+1} - k - 2$$

Substituting back into (3.5), we get

$$\begin{aligned}(2/v)^{2^k-1}2^{-\sum_{i=1}^{k} i2^{k-i}} &= (2/v)^{2^k-1}2^{k+2-2^{k+1}} \\ &= (\frac{1}{v})^{2^k-1}2^{2^k-1}2^{k+2-2\cdot 2^k} \\ &= v^{1-2^k}2^{-2^k+k+1}\end{aligned} \tag{3.6}$$

Substituting (3.6) back into (3.4) gives

$$\|[\tau_t(A), B]\| \leq \frac{2\|A\|\|B\|\|F_a\|}{C_{F_\mu}} \left(v^{1-2^k}2^{-2^k+k+1}\right)(e^{v|t|} - P_{2^k}(v|t|))e^{-\mu n/2^k} \tag{3.7}$$

We are interested in finding v, k, and n so that (3.7) is a better bound than both the traditional and short-range interaction Lieb-Robinson bounds. That is, we wish to have

$$\begin{aligned}&v^{1-2^k}2^{-2^k+k+1}(e^{v|t|} - P_{2^k}(v|t|))e^{-\mu n/2^k} \leq (e^{v|t|} - P_{2^k}(v|t|))e^{-\mu n} \\ \Rightarrow &v^{2^k-1} \leq e^{\mu n(1-1/2^k)}2^{-2^k+k+1} \\ \Rightarrow &v^{2^k-1} \leq \left[e^{\mu n/2^k}2^{(-2^k+k+1)/(2^k-1)}\right]^{2^k-1} \\ \Rightarrow &v \leq e^{\mu n/2^k}2^{(-2^k+k+1)/(2^k-1)}.\end{aligned} \tag{3.8}$$

Recalling that $n = 2^p$, we can simplify further to get

$$v \leq e^{\mu(2^{p-k})}2^{k/(2^k-1)-1} = e^{\mu 2^{p-k}+\ln(2)(k/(2^k-1)-1)}.$$

Now, if we study the exponent of (3.8) we see that since $p > k$ the exponent is positive which implies v is bounded above by a large value. Therefore, (3.7) is a better estimate for a large range of v values. In Figure 4, we plot the behavior of this system with the parameters set to match those in Figure 1. Comparing the two, it is apparent that accounting for the individual sizes of the Ψ_i increases the range of acceptable times by about two time-units.

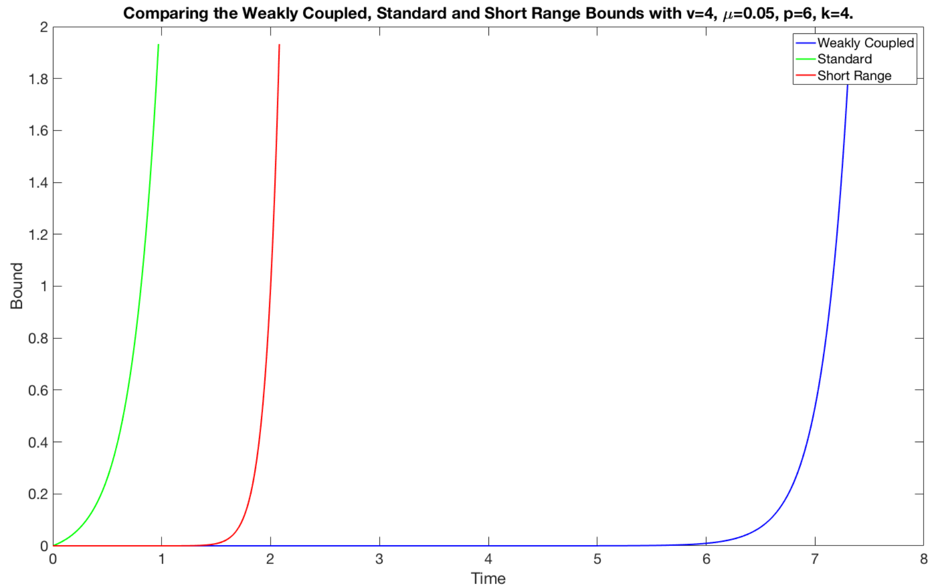

Figure 4: Comparing the standard and short-range Lieb-Robinson (LR) bounds with the estimate provided in Proposition 12 for a weakly coupled system of $N = 16$ identical intervals of width $n = 4$. Here, the interval LR-velocity is $v = 4$, $\mu = 0.05$, and each coupling is taken to be the strength determined in the setup of section 3.4. Notice that by considering each Ψ_i as opposed to taking the maximum $\|\Psi\|$ as in Figure 1, the range of acceptable times is extended by about two time-units.

3.5 Strongly Coupled Intervals

We can also consider a many-body system that is strongly coupled. If there is at least one coupling of greater strength than v, that is $\|\Psi\| := \max\{\|\Psi_i\|\} > v$, then the Lieb-Robinson velocity of the entire system becomes $2\|\Psi\|_{F_\mu} C_{F_\mu} := V > v$. Proposition 12 will exhibit improved behavior over the standard and short-range interaction Lieb-Robinson bounds when

$$\left(\prod_{i=1}^{N-1} \frac{2\|\Psi_i\|}{v}\right)(e^{v|t|} - P_{N-1}(v|t|))e^{-\mu n} \leq (e^{V|t|} - P_{N-1}(V|t|))e^{-\mu N n} \leq (e^{V|t|} - 1)e^{-\mu N n} \tag{3.9}$$

This certainly holds if

$$\|\Psi\|^{N-1} \leq \frac{e^{V|t|} - P_{N-1}(V|t|)}{e^{v|t|} - P_{N-1}(v|t|)} \left(\frac{ve^{-\mu n}}{2}\right)^{N-1} \tag{3.10}$$

since $\prod_{i=1}^{N-1} \|\Psi_i\| \leq \|\Psi\|^{N-1}$. It is not immediately clear which time regimes satisfy the inequality. However, we observe that as $|t| \to \infty$, the left hand side of (3.9) goes as $e^{v|t|}$ while the right hand side of (3.9) goes as $e^{V|t|}$. Thus, (3.9) certainly holds for large times. While the condition (3.10) is sufficient, it may be far from necessary. If for each strong coupling there exists is a corresponding weak coupling, the product $\prod_{i=1}^{N-1} \|\Psi_i\|$ may actually be on the order of $(\|\Psi\|/\alpha)^{N-1}$, where $\alpha > 1$. We plot this behavior in Figure 5, with $v = 1$, $V = 8$, $\mu = 0.5$, $\alpha = 1.5$ with maximum coupling strength of $\|\Psi\| = 4$ for a system of $N = 10$ intervals of width $n = 4$.

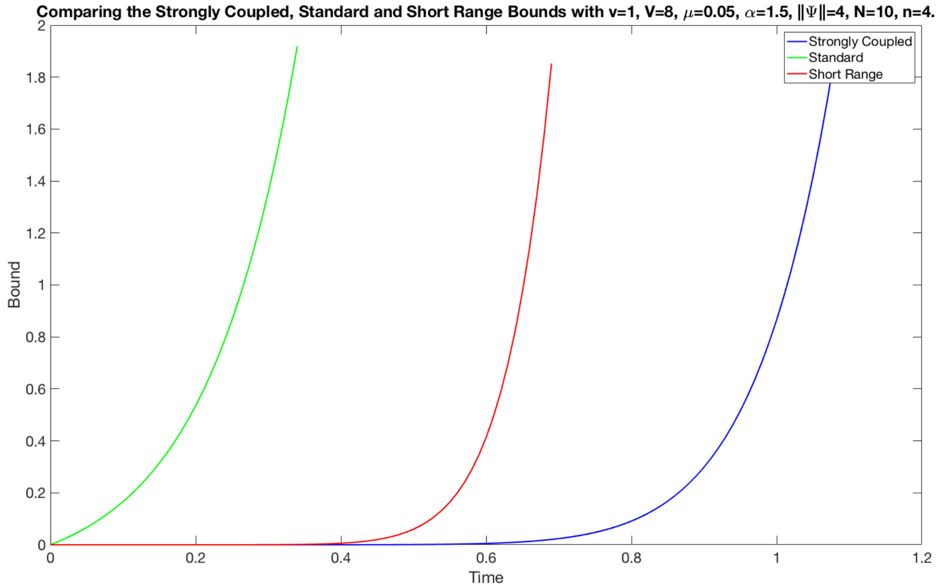

Figure 5: Comparing the standard and short-range Lieb-Robinson (LR) bounds with the estimate provided in Proposition 12 for a strongly coupled system of $N = 10$ identical intervals of width $n = 4$. Here, the component-interval LR-velocity is $v = 4$, the LR-velocity on Λ is $V = 8$, the average interaction strength $\|\Psi/\alpha\| = 2.67$, and $\mu = 0.05$

Again, we see that Proposition 12 implies greater localization than both Theorems 3 or 7.

3.6 Concluding Remarks

The Lieb-Robinson bound demonstrates the existence of a finite propagation velocity throughout quantum spin-systems. This finite velocity imposes a locality property on the lattice, essentially bounding how the support of observables spreads under time evolution.

In section 2, we derive a Lieb-Robinson bound for a spin chain defined on the lattice Λ, which depends only on the size of observables A and B, the interaction boundaries of their supports X and Y, and the interaction itself. In our derivation, we split the dynamics into forward and backwards time evolution, from which we obtain a differential equation describing how $[\tau_t^\Lambda(\tau_{-t}^X(A)), B]$ evolves in time. Capitalizing on the norm preserving property of $[iH(t), \cdot\ ]$, we obtain an initial integral bound on the normed commutator $\|[\tau_t^\Lambda(A), B]\|$. Successively approximating the integrands of these integral bounds leads to a series bound discussed in section 2.4. In section 2.4.1 we make use of an $\mathfrak{F}$-function to reduce the series terms which ultimately leads to the expression for the Lieb-Robinson velocity of the system. If we use the F_α form of the $\mathfrak{F}$-function, the final bound demonstrates in time regimes where $v|t| - \alpha d(X, Y) < 0$, the support of $\tau_t^\Lambda(A)$ remains localized from the support of B up to exponentially decaying error.

In section 2.5, we work with the premise that the interactions defining the dynamics are short-range. With this assumption, we develop a general bound accounting for the limited interaction reach. The result is a better estimate to the standard Lieb-Robinson bound which holds for all time.

In section 3, we turn to the interaction picture (section 1.4) to study the behavior of coupled systems. The interaction picture provides a method to split off the well behaved portions of the dynamics and address the remaining components separately. We show in section 3.1 that coupled systems can be bounded by a Lieb-Robinson like bound given in terms of the minimum component velocity, at the cost of polynomial factors of time. Further work could improve this bound by explicit integration methods (as was done in section 3.4), although extending this to a general case is much less tractable for randomly arranged velocities. Continuing, we show in 3.2 that three interval systems which are linked by a zero-velocity interval exhibit strong localization and the bound on $\|[\tau_t(A), B]\|$ goes as a quadratic in time as opposed to the standard exponential behavior.

We continue to investigate coupled systems in 3.3. We show that chains consisting of linked identical intervals, each with a common Lieb-Robinson velocity v, are bound by short-range Lieb-Robinson like expression with an added pre-factor $\prod_{i=1}^{N-1} \frac{\|\Psi_i\|}{v}$, where N is the number of linked intervals and Ψ_i are the coupling interactions (see Proposition 12). In the case of weakly coupled intervals, the pre-factor decays, improving the bound over both the standard and short-range Lieb-Robinson estimates. Similarly, a strongly coupled system provides an improvement because the Lieb-Robinson velocity for the whole chain is increased by the strong couplings.

A Appendix

A.1 Proof of Lemma 4

Let $A(t)$, for $t \in I \subset \mathbb{R}$, be a family of norm preserving operators on a finite dimensional normed space X. For any continuous function $b : I \to X$, the solution of,

$$\partial_t y(t) = A(t)y(t) + b(t) \tag{A.1.1}$$

with initial condition $y(t_0) = y_0$, satisfies the bound

$$\|y(t) - T(t)(y_0)\| \leq \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \|b(s)\|ds \tag{A.1.2}$$

where T is the solution operator such that $f(t) = T(t)f_0$ is the solution to the homogeneous form of (A.1.1).

Proof. By Duhamel's Principle, we know if $T : X \to X$ is the mapping such that $f(t) = T(t)f_0$, $f(t_0) = f_0$, satisfies $f'(t) = A(t)f(t)$, then

$$y(t) = T(t)(y_0) + \int_{t_0}^{t} T(t-s)b(s)ds \tag{A.1.3}$$

is the solution to (A.1.1). Thus, we have that since $A(t)$ is norm preserving,

$$\begin{aligned} \|y(t) - T(t)(y_0)\| &= \| \int_{t_0}^{t} T(t-s)b(s)ds\| \\ &\leq \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \|T(t-s)b(s)\|ds \\ &= \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} \|b(s)\|ds \end{aligned}$$

since $\|T(t)y_0\| = \|y_0\|$ for all $t \in \mathbb{R}$ and all $y_0 \in X$. Therefore, (A.1.2) holds as desired. □

A.2 Demonstrating $[iH(t), \cdot\ ]$ is a Norm Preserving Operator

Here we show in general that the family of operators $i[H(t), \ \cdot\ ]$ is norm preserving, where $H(t)$ is some time dependent self adjoint Hamiltonian.

Proof. Consider the initial value problem

$$\frac{dx}{dt} = i[H(t), x] \qquad x(0) = x_0, \tag{A.2.1}$$

where $x : \mathbb{R} \to \mathbb{C}^d$, for some $d \in \mathbb{N}$ and $H : \mathbb{R} \to M_d(\mathbb{C})$ is a bounded self-adjoint operator. We will show that $[H(t), \cdot\ ]$ is norm preserving.

Suppose $x(t) = U^*(t)x_0U(t)$, and that $U(t)$ is the unique solution to the IVP

$$\frac{dU}{dt} = -iU(t)H(t), \qquad U(0) = \mathbb{1} \tag{A.2.2}$$

Note that we know this solution exists in the form of the absolutely convergent *Dyson series*

$$U(t) = \mathbb{1} + \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} i^k H(s_1)H(s_2)\cdots H(s_k) ds_k ds_{k-1} \cdots ds_1.$$

Then we have that $\frac{d(U^*)}{dt} = \left(\frac{dU}{dt}\right)^* = iH(t)U^*(t)$, which implies

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}[U^*(t)]x_0U(t) + U^*(t)x_0\frac{d}{dt}[U(t)] \\ &= iH(t)U^*(t)x_0U(t) - iU^*(T)x_0U(t)H(t) \\ &= i[H(t), x(t)]\end{aligned}$$

so $x(t) = U^*(t)x_0U(t)$ solves (A.2.1) since $x(0) = U^*(0)x_0U(0) = x_0$.
We further claim that $U(t)$ is unitary. Certainly we have that $U(0)U^*(0) = \mathbb{1}$. Consider further that

$$\begin{aligned}\frac{d}{dt}[UU^*] &= \frac{dU}{dt}U^* + U\frac{dU^*}{dt} \\ &= -iU(t)H(t)U^*(t) + iU(t)H(t)U^*(t) \\ &= 0\end{aligned}$$

So, UU^* is constant for all time, which implies $UU^* \equiv \mathbb{1}$ and, consequently, that $U^*U = \mathbb{1}$. Thus, we have that $x(t) = U^*(t)x_0U(t)$, $U(0) = \mathbb{1}$, is the unique solution to (A.2.1) and $\|x(t)\| = \|U^*(t)x_0U(t)\| = \|x_0\|$ because U is unitary. Therefore, the operator $[H(t), \cdot\ ]$ is norm preserving. □

A.3 Proof of Proposition 6

To show Proposition 6, we must show that for each $N \geq 1$,

$$\|[\tau_t^\Lambda(A), B]\| \leq 2\|A\|\|B\| \left(\delta_Y(X) + \sum_{n=1}^{N} a_n(t)\right) + R_{N+1}(t) \tag{A.3.1}$$

where

$$a_n(t) = \frac{(2|t|)^n}{n!} \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_n\in S_\Lambda(Z_{n-1})} \delta_Y(Z_n) \prod_{j=1}^{n} \|\Phi(Z_j)\| \tag{A.3.2}$$

and

$$R_{N+1}(t) = \frac{(2|t|)^{N+1}}{(N+1)!}\|A\| \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_{N+1}\in S_\Lambda(Z_N)} \sup_{s\in[0,|t|]} \|[\tau_s^\Lambda(\Phi(Z_{N+1})), B]\| \times \prod_{j=1}^{N} \|\Phi(Z_j)\|. \tag{A.3.3}$$

Before beginning the proof, it is useful to note that for all $t, s_1, \cdots, s_N \in \mathbb{R}$, the expression

$$\int_{\min\{0,t\}}^{\max\{0,t\}} \int_{\min\{0,s_1\}}^{\max\{0,s_1\}} \cdots \int_{\min\{0,s_{N-1}\}}^{\max\{0,s_{N-1}\}} \mathrm{d}s_N \ldots \mathrm{d}s_1$$

gives the volume of the N dimensional simplex, $\frac{|t|^N}{N!}$. Also for notational ease, throughout this proof we will denote the iterated integral

$$\int_{\min\{0,t\}}^{\max\{0,t\}} \int_{\min\{0,s_1\}}^{\max\{0,s_1\}} \cdots \int_{\min\{0,s_{N-1}\}}^{\max\{0,s_{N-1}\}} \mathrm{d}s_N \ldots \mathrm{d}s_1$$

by

$$\int_{m(0;t,s_1,\ldots,s_{N-1})}^{M(0;t,s_1,\ldots,s_{N-1})} \mathrm{d}s_N \cdots \mathrm{d}s_1,$$

where $M(0;t,s_1,\ldots,s_N) = \max\{0,t\},\ \max\{0,s_1\},\ldots,\max\{0,s_N\}$ and $m(0;t,s_1,\ldots,s_N) = \min\{0,t\},\ \min\{0,s_1\},\ldots,\min\{0,s_N\}$. Further, we denote the set $S_\Lambda(Z_i)$ by S_i, for $i = 0,1,\cdots$, where we let $Z_0 := X$.

Proof. (by induction) To show (A.3.1), we will need to induct. However, since we do not have a bound on the term $\sup_{s\in[0,|t|]} \|\tau_s^\Lambda(\Phi(Z_{N+1})), B]\|$ in $R_N(t)$, we will induct on a modified remainder expression given by

$$R'_{N+1}(t) = \|A\| 2^{N+1} \sum_{Z_1\in S_0} \cdots \sum_{Z_{N+1}\in S_N} \prod_{j=1}^{N+1} \|\Phi_j(Z_{N+1})\| \int_{m(0;t,s_1,\ldots,s_N)}^{M(0;t,s_1,\ldots,s_N)} C_B^\Lambda(Z_{N+1};s_{N+1}) \mathrm{d}s_{N+1}\cdots \mathrm{d}s_1. \tag{A.3.4}$$

That is, we will show that (A.3.1) holds, with $R'_{N+1}(t)$ replacing $R_{N+1}(t)$, and then show how this result implies Prop 6. We start with the base case $N = 1$.

Considering the inequality derived in Proposition 5, rewritten below,

$$C_B^\Lambda(X;t) \le 2||B||\delta_Y(X) + 2\sum_{Z_1\in S_0} ||\Phi(Z_1)|| \int_{m(0;t)}^{M(0;t)} C_B^\Lambda(Z_1;s_1)\mathrm{d}s_1, \tag{A.3.5}$$

we immediately have by replacing $C_B^\Lambda(Z_1;s_1)$ by its upper bound given in (2.13) that

$$\begin{aligned}
\frac{\|\tau_t^\Lambda(A),B]\|}{\|A\|} \le{}& C_B^\Lambda(X;t) \\
\le{}& 2\|B\|\delta_Y(X) + \\
& 2\sum_{Z_1\in S_0}\|\Phi(Z_1)\| \int_{m(0;t)}^{M(0;t)} \left(2\|B\|\delta_Y(Z_1) + 2\sum_{Z_2\in S_1}\|\Phi(Z_2)\| \int_{m(0;s_1)}^{M(0;s_1)} C_B^\Lambda(Z_2;s_2)\mathrm{d}s_2 \right)\mathrm{d}s_1 \\
={}& 2\|B\|\delta_Y(X) + 2^2\|B\|\sum_{Z_1\in S_0}\|\Phi(Z_1)\|\delta_Y(z_1)\int_{m(0;t)}^{M(0;t)}\mathrm{d}s_1 + \\
& 2^2\sum_{Z_1\in S_0}\sum_{Z_2\in S_1}\|\Phi(Z_1)\|\|\Phi(Z_2)\|\int_{m(0;t,s_1)}^{M(0;t,s_1)} C_B^\Lambda(Z_2;s_2)\mathrm{d}s_2\mathrm{d}s_1 \\
\le{}& 2\|B\|\left(\delta_Y(X) + 2|t|\sum_{Z_1\in S_0}\|\Phi(Z_1)\|\delta_Y(Z_1)\right) + \\
& 2^2\sum_{Z_1\in S_0}\sum_{Z_2\in S_1}\|\Phi(Z_1)\|\|\Phi(Z_2)\|\int_{m(0;t,s_1)}^{M(0;t,s_1)} C_B^\Lambda(Z_2;s_2)\mathrm{d}s_2\mathrm{d}s_1
\end{aligned}$$

$$=2\|B\|\left(\delta_Y(X)+a_1(t)\right)+R_2'(t)/\|A\|. \tag{A.3.6}$$

Multiplying (A.3.6) by $\|A\|$ gives the base case.

Now suppose that (A.3.1) (with R' exchanged for R) holds for some $N\in\mathbb{N}$. That is, suppose

$$\|[\tau_t^\Lambda(A),B]\| \le 2\|A\|\|B\|\left(\delta_Y(X)+\sum_{n=1}^{N}a_n(t)\right)+ \\ \|A\|2^{N+1}\sum_{Z_1\in S_0}\cdots\sum_{Z_{N+1}\in S_{Z_N}}\prod_{j=1}^{N+1}\|\Phi(Z_j)\|\int_{m(0;,t,s_1,\ldots,s_N)}^{M(0;t,s_1,\ldots,s_N)} C_B^\Lambda(Z_{N+1};s_{N+1})\mathrm{d}s_{N+1}\ldots\mathrm{d}s_1. \tag{A.3.7}$$

The inductive step follows by applying (2.13) to $C_B^\Lambda(Z_{N+1};s_{N+1})$ in (A.3.7), as shown below.

$$\int_{m(0;,t,s_1,\ldots,s_N)}^{M(0;t,s_1,\ldots,s_N)} C_B^\Lambda(Z_{N+1};s_{N+1})\mathrm{d}s_{N+1}\ldots\mathrm{d}s_1 \\ \le \int_{m(0;,t,s_1,\ldots,s_N)}^{M(0;t,s_1,\ldots,s_N)} 2\left(\|B\|\delta_Y(Z_{N+1})+\sum_{Z_{N+2}\in S_{N+1}}\|\Phi(Z_{N+2})\|\int_{m(0;s_{N+2})}^{M(0;s_{N+2})} C_B^\Lambda(Z_{N+2};s_{N+2})\mathrm{d}s_{N+2}\right)\mathrm{d}s_{N+1}\cdots\mathrm{d}s_1 \\ =\frac{2|t|^{N+1}}{(N+1)!}\|B\|\delta_Y(Z_{N+1})+2\sum_{Z_{N+2}\in S_{N+1}}\|\Phi(Z_{N+2})\|\int_{m(0;s_1,\ldots,s_{N+1})}^{M(0;t,s_1,\ldots,s_{N+1})} C_B^\Lambda(Z_{N+2};s_{N+2})\mathrm{d}s_{N+2}\cdots\mathrm{d}s_1. \tag{A.3.8}$$

Replacing the iterated integral in (A.3.7) by (A.3.8) will give

$$\|[\tau_t^\Lambda(A),B]\| \le 2\|A\|\|B\|\left(\delta_Y(X)+\sum_{n=1}^{N}a_n(t)\right)+2\|A\|\|B\|a_{N+1}(t)+ \\ \|A\|2^{N+2}\sum_{Z_1\in S_0}\ldots\sum_{Z_{N+2}\in S_{Z_{N+1}}}\prod_{j=1}^{N+2}\|\Phi(Z_j)\|\int_{m(0;,t,s_1,\ldots,s_{N+1})}^{M(0;t,s_1,\ldots,s_{N+1})} C_B^\Lambda(Z_{N+2};s_{N+2})\mathrm{d}s_{N+2}\ldots\mathrm{d}s_1 \\ =2\|A\|\|B\|\left(\delta_Y(X)+\sum_{n=1}^{N+1}a_n(t)\right)+R_{N+2}'(t). \tag{A.3.9}$$

This establishes that (A.3.7) holds for all $N\in\mathbb{N}$ by induction.

Now we will show that (A.3.7) implies (A.3.1). Observe from the proof of Proposition 5 that for any $Z\subseteq\Lambda$,

$$C_B^\Lambda(Z;s)=\sup_{\substack{K\in\mathcal{A}_Z\\K\neq 0}}\frac{\|\tau_s^\Lambda(K),B]\|}{\|K\|}=\sup_{\substack{K\in\mathcal{A}_Z\\K\neq 0}}\frac{\|\tau_s^\Lambda(\tau_{-s}^Z(K)),B]\|}{\|K\|} \\ \le 2\|B\|\delta_Y(Z)+2\sum_{Z'\in S_\Lambda(Z)}\int_{m(0;s)}^{M(0;s)}\|[\tau_{s'}^\Lambda(\Phi(Z')),B]\|\mathrm{d}s', \tag{A.3.10}$$

where the inequality holds by 2.9.

Thus, by plugging (A.3.10) into $R'_N(t)$, we get that

$$\begin{aligned}
R'_N(t) =&\|A\|2^N \sum_{Z_1\in S_0}\cdots\sum_{Z_N\in S_{N-1}}\prod_{j=1}^{N}\|\Phi_j(Z_N)\|\int_{m(0;t,s_1,\dots,s_{N-1})}^{M(0;t,s_1,\dots,s_{N-1})} C_B^{\Lambda}(Z_N;s_N)\mathrm{d}s_N\cdots\mathrm{d}s_1\\
\leq&\|A\|2^N \sum_{Z_1\in S_0}\cdots\sum_{Z_N\in S_{N-1}}\prod_{j=1}^{N}\|\Phi_j(Z_N)\|\times\\
&\int_{m(0;t,s_1,\dots,s_{N-1})}^{M(0;t,s_1,\dots,s_{N-1})}\left(2\|B\|\delta_Y(Z_N)+2\sum_{Z_{N+1}\in S_{Z_N}}\int_{m(0;s_N)}^{M(0;s_N)}\|[\tau^{\Lambda}_{s_{N+1}}(\Phi(Z_{N+1})),B]\|\mathrm{d}s_{N+1}\right)\mathrm{d}s_N\cdots\mathrm{d}s_1\\
=&2\|A\|\|B\|\frac{(2|t|)^N}{N!}\sum_{Z_1\in S_0}\cdots\sum_{Z_N\in S_{N-1}}\prod_{j=1}^{N}\|\Phi_j(Z_N)\|\delta_Y(Z_N)+\\
&2^{N+1}\sum_{Z_1\in S_0}\cdots\sum_{Z_{N+1}\in S_{Z_N}}\prod_{j=1}^{N}\|\Phi_j(Z_N)\|\int_{m(0;t,s_1,\dots,s_N)}^{M(0;t,s_1,\dots,s_N)}\|[\tau^{\Lambda}_{s_{N+1}}(\Phi(Z_{N+1})),B]\|\mathrm{d}s_{N+1}\cdots\mathrm{d}s_1\\
\leq&2\|A\|\|B\|a_N(t)+\|A\|2^{N+1}\sum_{Z_1\in S_0}\cdots\sum_{Z_{N+1}\in S_{Z_N}}\prod_{j=1}^{N}\|\Phi_j(Z_N)\|\times\\
&\qquad\sup_{s\in[0,|t|]}\|[\tau^{\Lambda}_{s}(\Phi(Z_{N+1})),B]\|\int_{m(0;t,s_1,\dots,s_N)}^{M(0;t,s_1,\dots,s_N)}\mathrm{d}s_{N+1}\cdots\mathrm{d}s_1\\
=&2\|A\|\|B\|a_N(t)+\|A\|\frac{(2|t|)^{N+1}}{(N+1)!}\sum_{Z_1\in S_0}\cdots\sum_{Z_{N+1}\in S_{Z_N}}\prod_{j=1}^{N}\|\Phi_j(Z_N)\|\sup_{s\in[0,|t|]}\|[\tau^{\Lambda}_{s}(\Phi(Z_{N+1})),B]\|\\
=&2\|A\|\|B\|a_N(t)+R_{N+1}(t) \qquad\qquad (A.3.11)
\end{aligned}$$

So, we can take equation (A.3.7) and use (A.3.11) to replace $R'_N(t)$, which gives

$$\begin{aligned}
\tau_t^{\Lambda}(A),B]\| &\leq 2\|A\|\|B\|\left(\delta_Y(X)+\sum_{n=1}^{N-1}a_n(t)\right)+R'_N(t)\\
&\leq 2\|A\|\|B\|\left(\delta_Y(X)+\sum_{n=1}^{N-1}a_n(t)\right)+2\|A\|\|B\|\cdot a_N(t)+R_{N+1}(t) \qquad (A.3.12)\\
&\leq 2\|A\|\|B\|\left(\delta_Y(X)+\sum_{n=1}^{N}a_n(t)\right)+R_{N+1}(t)
\end{aligned}$$

Thus, we have that (A.3.7) implies (A.3.1) as desired.

□

A.3.1 Proving $R_N(t)$ has a vanishing limit

To finish the proof of Proposition 6, we use an $\mathfrak{F}$-function to bound $R_N(t)$ and then determine its limit. To begin, we assume that X has finite cardinality, and let $k = |X| < \infty$. First, we apply the trivial bound to the normed commutator in $R_N(t)$, to get

$$R_N(t) = \frac{(2|t|)^N}{N!}\|A\| \sum_{Z_1 \in S_0} \cdots \sum_{Z_N \in S_{N-1}} \sup_{s \in [0,|t|]} \|[\tau_s^\Lambda(\Phi(Z_N)), B]\| \prod_{j=1}^{N-1} \|\Phi(Z_j)\|$$
$$\leq 2\|A\|\|B\| \frac{(2|t|)^N}{N!} \sum_{Z_1 \in S_0} \cdots \sum_{Z_N \in S_{N-1}} \prod_{j=1}^{N} \|\Phi(Z_j)\|.$$

Then, similar to the argument made in section 2.4.1, we construct an over-count for $R_N(t)$ by

$$\begin{aligned}
R_N(t) &\leq 2\|A\|\|B\| \frac{(2|t|)^N}{N!} \sum_{Z_1 \in S_0} \cdots \sum_{Z_N \in S_{N-1}} \prod_{j=1}^{N} \|\Phi(Z_j)\| \\
&\leq 2\|A\|\|B\| \frac{(2|t|)^N}{N!} \sum_{x \in X} \sum_{\substack{w_1 \in \Lambda \\ w_2 \in \Lambda \\ \vdots \\ w_N \in \Lambda}} \sum_{\substack{Z_1 \subset \Lambda: \\ x \in Z_1 \\ w_1 \in Z_1}} \sum_{\substack{Z_2 \subset \Lambda: \\ w_1 \in Z_2 \\ w_2 \in Z_2}} \cdots \sum_{\substack{Z_N \subset \Lambda: \\ w_{N-1} \in Z_N \\ w_N \in Z_N}} \prod_{j=1}^{N} \|\Phi(Z_j)\| \\
&= 2\|A\|\|B\| \frac{(2|t|)^N}{N!} \sum_{x \in X} \sum_{\substack{w_1 \in \Lambda \\ w_2 \in \Lambda \\ \vdots \\ w_N \in \Lambda}} \sum_{\substack{Z_1 \subset \Lambda: \\ x \in Z_1 \\ w_1 \in Z_1}} \|\Phi(Z_1)\| \sum_{\substack{Z_2 \subset \Lambda: \\ w_1 \in Z_2 \\ w_2 \in Z_2}} \|\Phi(Z_2)\| \cdots \sum_{\substack{Z_N \subset \Lambda: \\ w_{N-1} \in Z_N \\ w_N \in Z_N}} \|\Phi(Z_N)\| \\
&\leq 2\|A\|\|B\| \frac{(2|t|)^N}{N!} \sum_{x \in X} \sum_{\substack{w_1 \in \Lambda \\ w_2 \in \Lambda \\ \vdots \\ w_N \in \Lambda}} \|\Phi\|_F F(|x - w_1|) \|\Phi\|_F F(|w_1 - w_2|) \cdots \|\Phi\|_F F(|w_{N-1} - w_N|) \\
&= 2\|A\|\|B\| \frac{(2\|\Phi\|_F |t|)^N}{N!} \sum_{x \in X} \sum_{w_N \in \Lambda} \sum_{w_{N-1} \in \Lambda} \cdots \sum_{w_1 \in \Lambda} F(|x - w_1|) F(|w_1 - w_2|) \cdots F(|w_{N-1} - w_N|) \\
&\leq 2\|A\|\|B\| \frac{(2\|\Phi\|_F |t|)^N}{N!} C_F \sum_{x \in X} \sum_{w_N \in \Lambda} \cdots \sum_{w_2 \in \Lambda} F(|x - w_2|) F(|w_2 - w_3|) \cdots F(|w_{N-1} - w_N|) \\
&\ \vdots \\
&\leq 2\|A\|\|B\| \frac{(2\|\Phi\|_F \ |t|)^N}{N!} C_F{}^{N-1} \sum_{x \in X} \sum_{w_N \in \Lambda} F(|x - w_N|) \\
&\leq 2\|A\|\|B\| \frac{(2\|\Phi\|_F \ C_F \|t|)^N}{N! C_F} \cdot k\|F\|, \qquad \text{(A.3.13)}
\end{aligned}$$

where the vertical dots denote the repeated application of the convolution property. Now, the right-hand-side of (A.3.13) is of the form $\lambda x^N/N!$. Since $\lim_{N\to\infty} x^N/N! = 0$ for any x, by comparison the expression in (A.3.13) also tends to zero as $N \to \infty$. Following the chain of inequalities, we conclude that $\lim_{n\to\infty} R_N(t) = 0$, for any $t \in \mathbb{R}$.

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