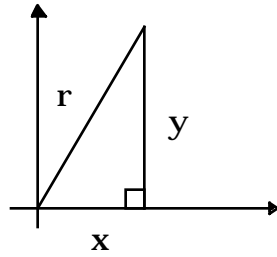


# Trigonometry Review

# Workshop 1

Definitions: Let P(x,y) be any point (not the origin) on the terminal side of an angle with measure  $\theta$  and let r be the distance from the origin to P. Then the six trig functions are defined as follows:



$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\csc \theta = \frac{r}{y} = \frac{1}{\sin \theta}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

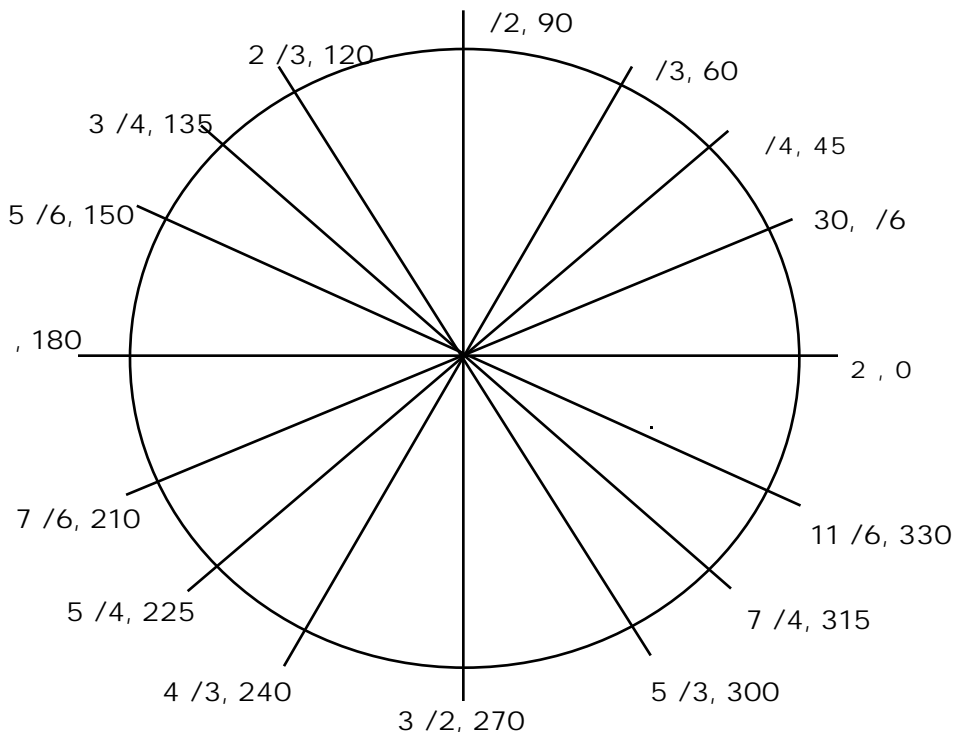
$$\sec \theta = \frac{r}{x} = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{x}{y} = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

## Measuring Angles

There are  $360^\circ$  in a circle and  $2\pi$  radians. They are marked off on the circle below:

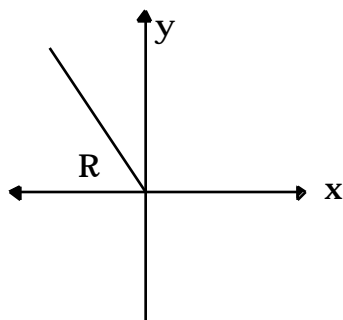


To convert from degrees to radians (and vice versa) we use the fact that  $180^\circ = \pi$  radians.

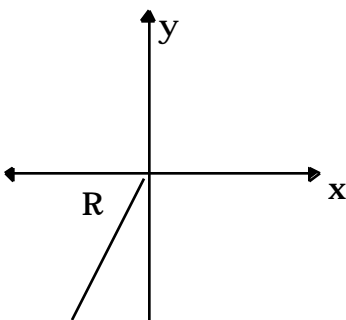
**Example:** Convert  $36^\circ$  to radians.

Using the ratio  $\frac{\text{radians}}{180^\circ}$  we find that  $36^\circ = 36^\circ \frac{\text{radians}}{180^\circ} = \frac{2}{10} = \frac{1}{5}$  radians

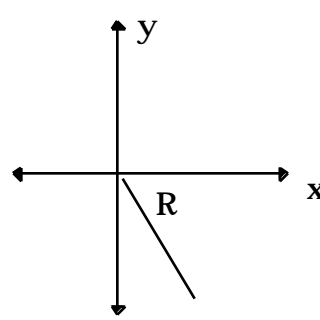
Reference Angles The angle formed by the terminal side of the angle and the x-axis is called the reference angle. The six trig functions of any angle are determined by it's reference angle and the quadrant it is in.



$$R = 180^\circ -$$



$$R = -180$$



$$R = 360^\circ -$$

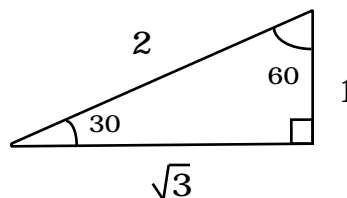
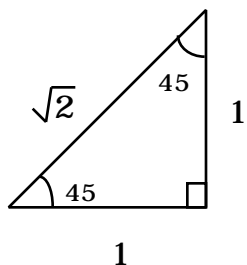
The signs of the three major trig functions are determined by the quadrant as follows:

- Quadrant I: All are positive
- Quadrant II: Sine is positive
- Quadrant III: Tangent is positive
- Quadrant IV: Cosine is positive

This can be remembered by the pneumonic device **All Students Take Calculus**. Since

$\csc = \frac{1}{\sin}$ , sine and cosecant always have the same sign. The is true of Cosine, Secant, and Tangent, Cotangent as well.

Special Triangles: We can use the ratios of some common triangles, along with the definitions of the trig. functions to determine the trig functions of some special angles. Using the two triangles below we see that:



$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{1}{1} = 1$$

$$\sin 30^\circ = \frac{1}{2}$$

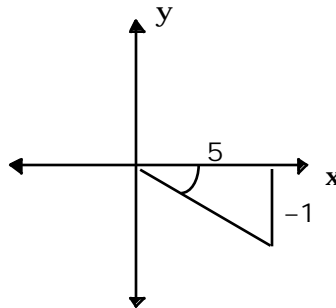
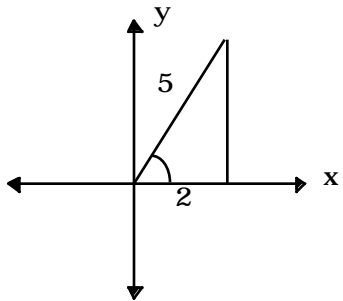
$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

and similarly for  $60^\circ$ . This is left as an exercise.

### Practice Exercises #1

1. Find the six trig functions of  $60^\circ$ .
2. Find the six trig functions for each of the following angles as shown:
  - a.
  - b.



3. Find the six trig functions of each of the following angles:

- a.  $135^\circ$

- b.  $\frac{5}{6}$

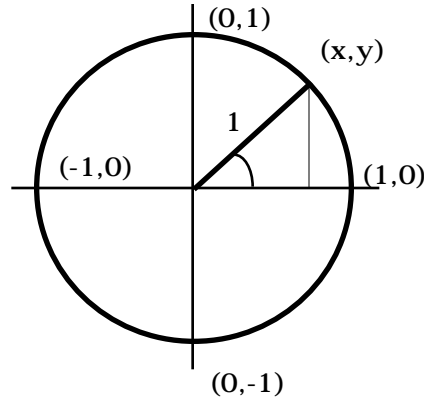
4. Given that  $\sin \theta = -\frac{1}{2}$  and  $\theta$  is in quadrant IV, find the cosine and tangent of  $\theta$ .
5. Convert  $124^\circ$  into radians.
6. Convert  $\frac{7}{8}$  radians into degrees.

## Trigonometry Review

## Workshop 2

Last time we saw the definitions of the six trig. functions and evaluated them for some special angles. There are a few more "nice" angles which we will discuss here.

Another way to define the functions  $\sin$  and  $\cos$  is as follows. If we look at the unit circle below,



and let  $(x,y)$  be any point on the circle, then  $\cos = x$ , and  $\sin = y$ . To see this simply examine the triangle above and use the previous definitions of these functions.

Thus to find the sines and cosines of angles we only need to find the x and y components of the point on the unit circle that corresponds to that angle. Hence we see that:

$$\begin{array}{cccc} \cos 0^\circ = 1 & \cos 90^\circ = 0 & \cos 180^\circ = -1 & \cos 270^\circ = 0 \\ \sin 0^\circ = 0 & \sin 90^\circ = 1 & \sin 180^\circ = 0 & \sin 270^\circ = -1 \end{array}$$

Since we know the sines and cosines for these angles we can find values for the other six trig functions:

Fill in these for  $0^\circ$  and  $90^\circ$  below:

$$\tan 0^\circ = \underline{\hspace{2cm}} \qquad \tan 90^\circ = \underline{\hspace{2cm}}$$

$$\cot 0^\circ = \underline{\hspace{2cm}} \qquad \cot 90^\circ = \underline{\hspace{2cm}}$$

$$\sec 0^\circ = \underline{\hspace{2cm}} \qquad \sec 90^\circ = \underline{\hspace{2cm}}$$

$$\csc 0^\circ = \underline{\hspace{2cm}} \qquad \csc 90^\circ = \underline{\hspace{2cm}}$$

At this point you should be able to calculate the trig functions of all angles with reference angles of  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , as well as all multiples of  $90^\circ$ . For other angle it will be necessary to use a calculator. Don't forget to put your calculator in the appropriate mode (either degrees or radians) depending on which you are using.

## Trigonometric Identities:

The following are some of the major trig identities which you will need to know. This list is NOT a complete list of all trig identities, simply the important ones.

### Pythagorean Identities:

$$\sin^2 + \cos^2 = 1$$

$$1 + \cot^2 = \csc^2$$

$$\tan^2 + 1 = \sec^2$$

(Note that the last two can be derived from the first by dividing by either  $\sin^2$  or  $\cos^2$ )

### Addition formulas:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

(Note that for the cosine addition formula

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

the +/- are reversed)

If we let  $A=B$  in the above formulas and choose the "+" we get the

### Double Angle Identities:

$$\sin(2A) = 2\sin A \cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

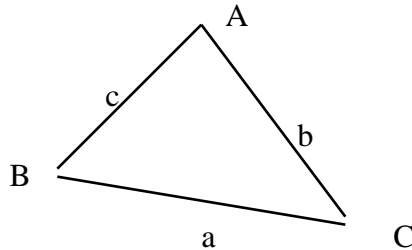
In addition we have the following laws which apply to triangles:

### Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

### Law of Sines:

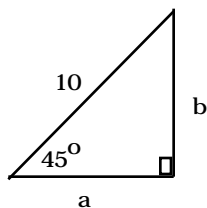
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



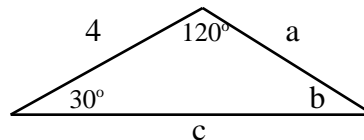
### **Practice Exercises #2:**

1. Find all sides and angles in the following triangles:

a.



b.



2. If  $\sin(A) = 1/3$  and  $A$  is in quadrant II, find: (a)  $\sin(2A)$ ; (b)  $\cos(2A)$ ; (c)  $\tan(2A)$

3. Solve for all values of  $0 < x < 2\pi$  that satisfy: (a)  $\sin x = \frac{\sqrt{3}}{2}$  (b)  $2\cos x \sin x = \sin x$

4. Verify the identity  $\sin^2 + \cos^2 = 1$  for  $\theta = 120^\circ$ .

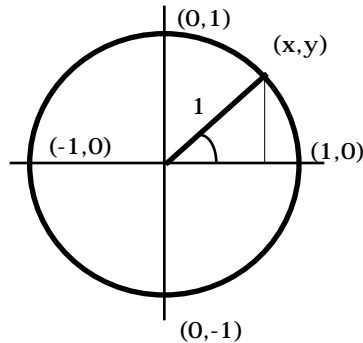
5. Find the six trig functions of  $180^\circ$

## Trigonometry Review

## Workshop 3

This time we will look at the graphs of  $y = \sin(x)$  and  $y = \cos(x)$ , as well as some variations.

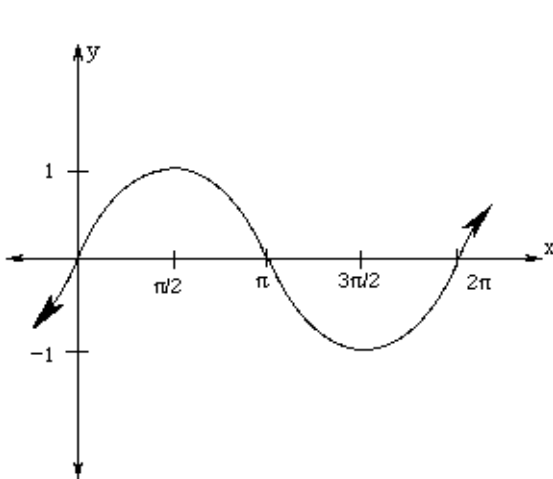
Recall last time we defined the functions  $\sin$  and  $\cos$  as follows. If we look at the unit circle below,



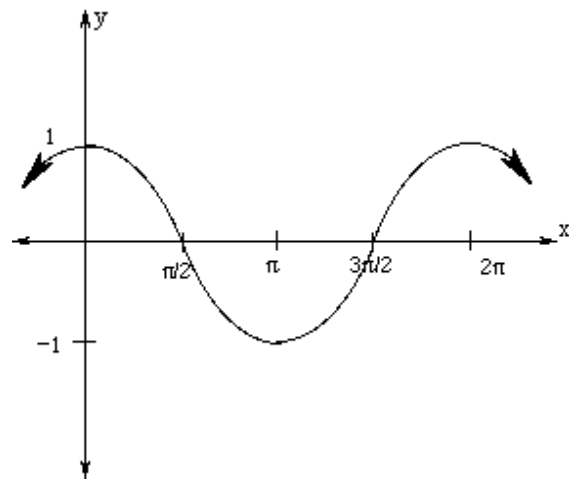
and let  $(x, y)$  be any point on the circle, then  $\cos = x$ , and  $\sin = y$ . Notice that as we move around the circle  $y = \sin(x)$ , which measures the  $y$  value of the point starts at 0 and increases to 1 at  $x = \pi/2$ . Cosine however, which measures the  $x$  value of the point starts at 1 then decreases to 0 at  $x = \pi/2$ . They are both described in the tables below:

$x$	$y = \sin(x)$	$y = \cos(x)$
0	0	1
$\pi/2$	1	0
$\pi$	0	-1
$3\pi/2$	-1	0
$2\pi$	0	1

Thus we have the following graphs of  $y = \sin(x)$ , and  $y = \cos(x)$  respectively:



$y = \sin(x)$

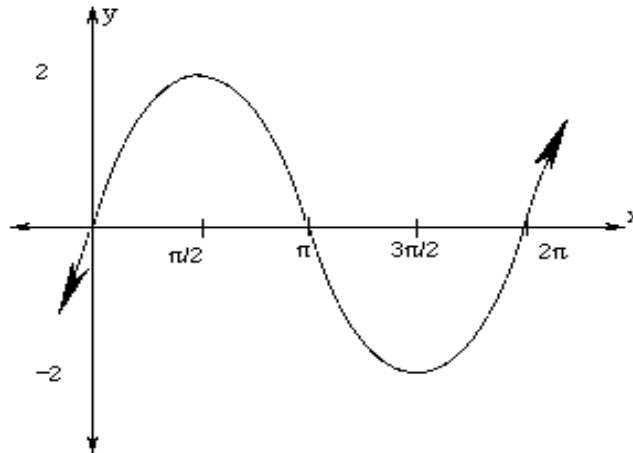


$y = \cos(x)$

Now what happens if we change things a bit? For example, let's look at the graph of  $y = 2\sin(x)$ . If we complete the chart as we did above we have:

$x$	$y = \sin(x)$	$y = 2\sin(x)$
0	0	0
$\pi/2$	1	2
$\pi$	0	0
$3\pi/2$	-1	-2
$2\pi$	0	0

and the graph:

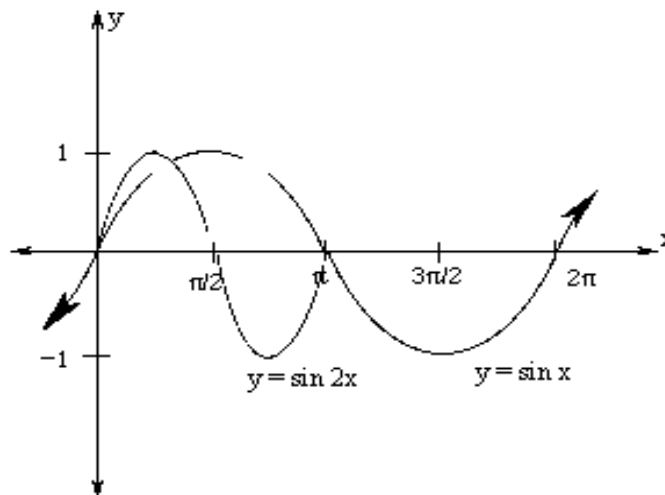


The graph has effectively been stretched vertically. This stretch changes the *amplitude* of the graph.

**The Amplitude of  $y = A\sin(x)$  is  $|A|$ .**

Now let's consider that graph of  $y = \sin(2x)$ . Notice that this differs from the above function in that the 2 is inside that sine (not outside).

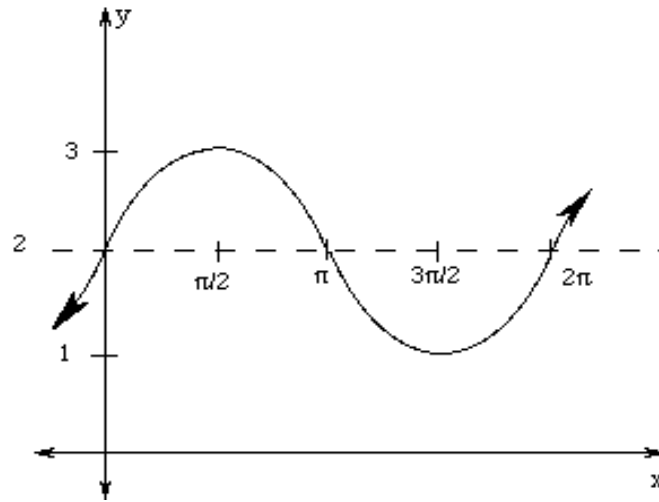
$x$	$2x$	$y = \sin(2x)$
0	0	0
$\pi/4$	$\pi/2$	1
$\pi/2$	$\pi$	0
$3\pi/4$	$3\pi/2$	-1
$\pi$	$2\pi$	0



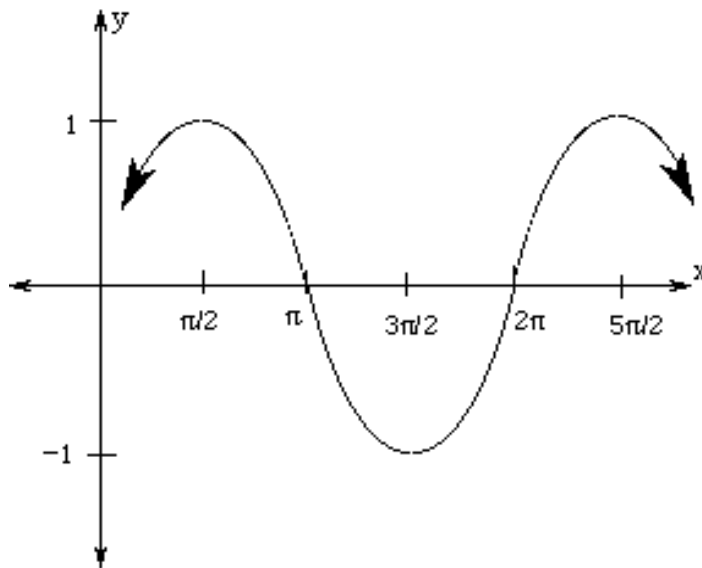
Notice that the graph has been "shrunk" horizontally by  $1/2$ . This changes the *period* of the function (or the distance before the function begins repeating itself).

**In general the period of  $y = \sin(Bx)$  is  $2\pi / B$ .**

Consider the graph of  $y = \sin(x)+1$ . Notice that we are simply adding one to the y-value for each x-value. This has the effect of shifting the graph vertically by one:



As our last example consider the function  $y = \cos(x - \pi/2)$ . Since we are subtracting from the x-value (not the y-value) this will shift the graph horizontally. Although it might seem reasonable to expect a shift to the left, the graph will be shifted to the right. The easiest way to determine the shift is to set the angle of the function to zero. In this case that means setting  $x - \pi/2 = 0$  and so  $x = \pi/2$ . Thus the shift is to the right by a positive  $\pi/2$ , and we have the graph:



**Practice Exercises #3:**

1. Find the period of each of the following: (a)  $y = \sin(5x)$  (b)  $y = \cos(x/2)$  (c)  $y = \sin(x)$
2. Graph each of the following on a separate axis.  
 (a)  $y = 2 \sin(4x)$ ; (b)  $y = \cos(2x)+1$ ; (c)  $y = 3 \sin(x + \pi/2)$
3. Find an example of a sine function which has an amplitude of 3, period of  $5\pi$  and passes through the point (0,4)



In this workshop we will continue to discuss solving trigonometric equations and further identities.

**Example:** Solve  $\sin^2 x = 1$  for all  $0 \leq x < 2\pi$ .

**Solution:** This problem is very similar to solving the equation  $y^2=1$  and should be thought of as similar. There are two ways to approach solving it. One is to take the square root of both sides. The other is to put everything on one side and factor. The latter provides a more general way so that is the approach that will be taken here.

$$\begin{aligned} \sin^2 x &= 1 \\ \sin^2 x - 1 &= 0 \\ (\sin x + 1)(\sin x - 1) &= 0. \end{aligned}$$

Hence we have  $\sin x = 1$  or  $\sin x = -1$ . Recall that the sine represents the vertical distance on the unit circle. By examining the unit circle we see that  $\sin x = 1$  when  $x = \pi/2$  and  $\sin x = -1$  when  $x = 3\pi/2$  radians. Hence, the solutions set is  $\{\pi/2, 3\pi/2\}$ . Notice that we do not include  $2\pi$ , since it is not in the interval  $0 \leq x < 2\pi$ .

**Example:** Solve  $\cos 2x = 1$ , where  $0 \leq x < 2\pi$ .

There are several ways to approach this problem. One possible way would be to use the double angle formula for cosine. Let's solve the problem this way, then consider a different approach.

$$\begin{aligned} \cos 2x &= 1 \\ \cos^2 x - \sin^2 x &= 1 \\ (1 - \sin^2 x) - \sin^2 x &= 1 \\ 1 - 2\sin^2 x &= 1 \\ -2\sin^2 x &= 0 \\ \sin^2 x &= 0 \\ \sin x &= 0. \end{aligned}$$

Since  $\sin x = 0$  at  $x = 0$  and  $x = \pi$ , our solution set is  $\{0, \pi\}$ .

There is, however, another approach to this problem. Since  $2x$  is the angle, let's solve for  $2x$  first. Notice that  $\cos 2x$  has a period of  $2\pi$ . Hence, to any solution we obtain, we can add the period of this function and this will give us another solution. In solving  $\cos 2x = 1$  notice that there is only one place where the cosine is zero and that is at  $0$  radians. Hence,  $2x=0$ , and thus  $x=0$ . Adding one period to this answer gives a second answer  $0 + 2\pi = 2\pi$ . Hence the solutions are  $x = 0$  and  $x = \pi$  radians. Notice that if we add  $2\pi$  again we get  $2\pi + 2\pi = 4\pi$ , which is outside the range of answers we want. The solution set  $\{0, \pi\}$  is complete. Notice that this agrees with the answer above.

**Example:** Solve  $\sin^2 3x = 1$  for  $0 \leq x < 2\pi$ .

**Solution:** First take the square root of both sides to obtain  $\sin 3x = \pm 1$ . That is,

$$\sin 3x = 1, \text{ or } \sin 3x = -1.$$

The  $\sin = 1$  at  $x = \pi/2$  and the  $\sin = -1$  at  $x = 3\pi/2$ . Hence we have

$$\begin{aligned} 3x &= \pi/2, \text{ and } 3x = 3\pi/2, \text{ or} \\ x &= \pi/6, \text{ and } x = \pi/2. \end{aligned}$$

Again, this is not a complete set of solutions. The period of  $\sin 3x$  is  $2\pi/3$  and this needs to be added to each solution until the values exceed  $2\pi$ . Hence another solution is  $\pi/6 + 2\pi/3 = 5\pi/6$ . Continuing this process, we obtain the complete set of solutions:

$$x \in \{\pi/6, \pi/2, 5\pi/6, 7\pi/6, 3\pi/2, 11\pi/6\}.$$

## More Identities

Trig identities play an important part in calculus. Being able to identify the major ones, especially the double angle formulas and the famous  $\sin^2 x + \cos^2 x = 1$  is very important. There are, however, many other useful identities which can be proven using definitions and existing identities. For example, it can be shown that  $\cos 2x + 2\sin^2 x = 1$ .

To show this, start with one side and try to reduce it to the other. For this example (and in general) it is generally easier to start with the more difficult side and try to simplify it. Start by using the double angle formula for cosine:

$$\begin{aligned}\cos 2x + 2\sin^2 x &= (\cos^2 x - \sin^2 x) + 2\sin^2 x, \\ &= \cos^2 x + (-\sin^2 x + 2\sin^2 x), \\ &= \cos^2 x + \sin^2 x, \\ &= 1.\end{aligned}$$

**Example:** Prove the identity  $(\sec x)(\cot x)(\csc x) = \csc^2 x$ .

**Solution:** Beginning with the left hand side, and using the basic definitions of the trig functions we find:

$$\begin{aligned}(\sec x)(\cot x)(\csc x) &= \frac{1}{\cos x} \frac{\cos x}{\sin x} \frac{1}{\sin x} \\ &= \frac{1}{\sin^2 x} = \csc^2 x.\end{aligned}$$

### Practice Exercises #4:

1. Solve each of the following trigonometric equations:

(a)  $\sin(2x) = 1/2$

(b)  $\sec^2(3x) = 2$

(c)  $\cos^2 x + \sin x = 1$

(d)  $\tan(4x) = \sqrt{3}$

2. Verify each of the following identities:

(a)  $(\sin x)(\sec x) = \tan x$

(b)  $(\tan x)(\sin 2x) = 2 \sin^2 x$

(c)  $\frac{\tan x}{1 - \cos^2 x} = \frac{\sec x}{\sin x}$

(d)  $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$ .

**Solutions to Problem Set #1:** 1.  $\sin 60^\circ = \frac{\sqrt{3}}{2}$ ,  $\sec 60^\circ = 2$ ,  $\cos 60^\circ = \frac{1}{2}$ ,  $\csc 60^\circ = \frac{2}{\sqrt{3}}$ ,  $\cot 60^\circ = \frac{1}{\sqrt{3}}$ ,  $\tan 60^\circ = \sqrt{3}$

2. (a)  $\sin = \frac{\sqrt{21}}{5}$ ,  $\csc = \frac{5}{\sqrt{21}}$ ,  $\cos = \frac{2}{5}$ ,  $\sec = \frac{5}{2}$ ,  $\tan = \frac{\sqrt{21}}{2}$ ,  $\cot = \frac{2}{\sqrt{21}}$ ,

(b)  $\sin = \frac{-1}{\sqrt{26}}$ ,  $\csc = -\sqrt{26}$ ,  $\cos = \frac{5}{\sqrt{26}}$ ,  $\sec = \frac{\sqrt{26}}{5}$ ,  $\tan = \frac{-1}{5}$ ,  $\cot = -5$ , 3.  $\sin(135) = \frac{\sqrt{2}}{2}$ ,  $\csc(135) = \sqrt{2}$ ,  $\cos(135) = -\frac{\sqrt{2}}{2}$ ,  $\sec(135) = -\sqrt{2}$ ,  $\tan(135) = -1$ ,  $\cot(135) = -1$ ;  $\sin(5/6) = 1/2$ ,  $\sec(5/6) = 2$ ,  $\cos(5/6) = -\sqrt{3}/2$ ,  $\sec(5/6) = -2/\sqrt{3}$ ,  $\tan(5/6) = -1/\sqrt{3}$ ,  $\cot(5/6) = -\sqrt{3}$  4.  $\cos = \frac{\sqrt{3}}{2}$ ,  $\tan = -\frac{1}{\sqrt{3}}$ , 5.  $124^\circ = \frac{31}{45}$  radians, 6.  $\frac{7}{8}$  radians =  $157.5^\circ$

**Solutions to Problem Set #2:**

(1) (a)  $a = b = 5\sqrt{2}$ ,  $c = 45^\circ$  (b)  $a = 4$ ,  $b = 30^\circ$ ,  $c = 4\sqrt{3}$

(2) (a)  $\frac{4\sqrt{2}}{9}$  (b)  $\frac{7}{9}$  (c)  $\frac{4\sqrt{2}}{7}$

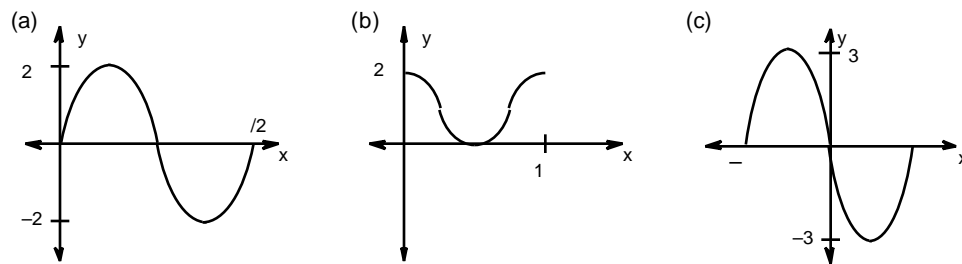
(3) (a)  $x = \frac{2}{3}, \frac{2}{3}$  (b)  $x = 0, \frac{5}{3}, \frac{5}{3}$

(4)  $\sin^2 120 + \cos^2 120 = (\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2 = \frac{3}{4} + \frac{1}{4} = 1$ .

(5)  $\sin(180^\circ) = 0$ ,  $\cos(180^\circ) = -1$ ,  $\tan(180^\circ) = 0$ ,  
 $\csc(180^\circ)$  is undefined,  $\sec(180^\circ) = -1$ ,  $\cot(180^\circ)$  is undefined.

**Solutions to Problem Set #3:**

1. (a)  $2/5$ , (b)  $4$ , (c)  $2$



3.  $y = 3\sin \frac{2}{5}x + 4$

**Solutions to Problem Set #4**

1. (a)  $x = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$  (b)  $x = \pi/12, 3\pi/12, 5\pi/12, 7\pi/12, 9\pi/12, 11\pi/12, 13\pi/12, 15\pi/12, 17\pi/12, 19\pi/12, 21\pi/12, 23\pi/12$ . (c)  $x = 0, \pi/2$ , (d)  $x = \pi/3, 5\pi/6, 4\pi/3, 11\pi/6$