MAT22A SECTION 2
Final Exam

Problem 1. (10 pts) Let

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 5 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) Compute the inverse of A.

(b) Use the inverse of A to solve the linear system of equations $Ax = b$ where

$$b = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

Problem 2. (10 pts) Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Is the matrix A in row reduced echelon form?

(b) Is the matrix B in row reduced echelon form?

(c) Find all solutions of the linear system $Ax = b$ where

$$b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

(d) Find all solutions of the linear system $Bx = c$ where

$$c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 3. (15 pts) Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

(a) Find a basis for the column space of A consisting of columns of the matrix A.

(b) Find a basis for the null space of A.

Problem 4. (15 pts) Let A be the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) What is the characteristic polynomial of A?

(b) Find all of the eigenvalues of A.

(c) For each eigenvalue λ of A find a basis for the vector space

$$V_\lambda = \{v : Av = \lambda v\}.$$

Problem 5. (10 pts) Find an **orthonormal** basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 6. (10 pts) Show that if the matrix A is similar to the matrix B then $\det(A) = \det(B)$. (reminder: two $n \times n$ matrices A and B are similar if there is an invertible $n \times n$ matrix P such that $A = P^{-1}BP$.)

Problem 7. (15 pts) Suppose that A is an 3×3 matrix such that

$$\begin{aligned} Av_1 &= 2w_1 - 2w_2 \\ Av_2 &= w_1 + w_2 - w_3 \\ Av_3 &= w_1 - w_3, \end{aligned}$$

where $S = \{v_1, v_2, v_3\}$ and $T = \{w_1, w_2, w_3\}$ are bases for $\mathbb{R}^3$. Suppose further that

$$\begin{aligned} w_1 &= v_1 - v_3 \\ w_2 &= v_2 - v_3 \\ w_3 &= v_1 + v_2. \end{aligned}$$

(a) What is the matrix of A with respect to the basis $\{v_1, v_2, v_3\}$?

(b) What is the matrix of A with respect to the basis $\{w_1, w_2, w_3\}$?

(c) What is the determinant of A?

Problem 8. (15 pts) Let $\mathbb{P}^n$ denote the vector space of polynomials of degree less than or equal to n, as usual. Consider the linear transformation $L : \mathbb{P}^3 \to \mathbb{P}^2$ by the formula

$$L(p(t)) = p'(t) - tp''(t).$$

where $p'(t)$ is the first derivative of the polynomial $p(t)$ and $p''(t)$ is the second derivative of the polynomial $p(t)$.

(a) Find the matrix of the linear transformation L with respect to the bases

$$S = \{1 + t^3, t + t^2, t^2 - t^3, t^3\}$$

and

$$T = \{1, t, t^2\}.$$

Note: S is a basis for $\mathbb{P}^3$ and T is a basis for $\mathbb{P}^2$, so S is the "input basis" and T the "output basis."

(b) Find a basis for the kernel of the transformation L. Write each of the vectors in the basis in terms of the canonical basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}^3$.

(c) Find a basis for the range of the transformation L. Write each of the vectors in the basis in terms of the canoncial basis $\{1, t, t^2\}$ for $\mathbb{P}^2$.