Fall 2003 Mathematics Graduate Program Preliminary Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Analysis

Problem 1. (a) For a function $f : (a,b) \to R^1$, (a,b) an open interval, state briefly but precisely:
i. What is meant by the statement: $f(x)$ is continuous at $x_0 \in (a,b)$.
ii. What is meant by the statement: $f(x)$ is continuous on (a,b).
iii. What is meant by the statement: $f(x)$ is uniformly continuous on (a,b).
(b) Prove, directly from the definition, that the function $f(x) = 1/x$ is uniformly continuous on the interval $[1,\infty)$.

Problem 2. Let $\{U_n\}_{n=1}^{\infty}$ be a nested sequence of open sets in a topological space X, so that $U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1}$. Let $x_n \in U_n \backslash U_{n-1}$ Set $U = \cup_{n=1}^{\infty} U_n$. Prove that $\{x_n\}$ does not have a subsequence that converges to a point in U.

Problem 3. Let $T : (X,d) \to (X,d)$ be a contraction mapping from the metric space (X,d) to itself, so that for some $r < 1$, $d(Tx,Ty) \leq rd(x,y)$ $\forall x,y \in X,$. Assume that x_0 is a fixed point of this mapping. Prove that

$$d(x,x_0) \leq \frac{d(x,T(x))}{1-r}$$

Problem 4. Let y, y' be two elements of a Hilbert space H.
Prove that if $< y,x >=< y',x >$ for every $x \in H$ then $y = y'$.

Problem 5. Let L and R be the left shift operator and the right shift operator of $l^2(\mathbb{N})$ respectively.So

$$L(x_1,x_2,x_3,\dots) = (x_2,x_3,x_4\dots)$$
$$R(x_1,x_2,x_3,\dots) = (0,x_1,x_2,x_3,x_4\dots).$$

Find the point specturm of L and R

Problem 6. Define the following three sequences of functions $[0,+\infty) \to \mathbb{R}$:

$(f_n)_{n=1}^{\infty}$ given by $f_n(x) = \begin{cases} \frac{n^{1/2}}{(x+1)^n} & \text{if } 0 \leq x \leq n \\ 0 & \text{else} \end{cases}$

$(g_n)_{n=1}^{\infty}$ given by $g_n(x) = \begin{cases} \sin(2\pi nx) & \text{if } n \leq x \leq n+1 \\ 0 & \text{else} \end{cases}$

and $(h_n)_{n=1}^{\infty}$ given by $h_n(x) = \sum_{k=1}^{n} \frac{k}{\sqrt{n}} \mathrm{Ind}_{[k,k+(1/n^2)]}(x)$.

Consider these sequences with each of the topologies given below and determine whether or not they converge and, if they converge, determine their limits. Explain your assertions.
a. Pointwise on $[0,+\infty)$.
b. Uniformly on $[0,+\infty)$.
c. In the norm topology of $L^2([0,+\infty))$.
d. Strongly in $L^{3/2}([0,+\infty))$.
e. Weakly in $L^{3/2}([0,+\infty))$.

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2. Algebra and Linear Algebra

Problem 7. Let G be a group and p a prime. Prove or give a counter example:
a. A group of order p is commutative.
b. A group of order p^2 is commutative.
c. A group of order p^3 is commutative.

Problem 8. Let F be a finite field. Show that the number of elements of F is p^r for some prime p and positive integer r.

Problem 9. A vector space V contains an n-element set with the following properties:
(i) It is not linearly independent, but contains an $(n-1)$-element linearly independent set;
(ii) It does not span V, but is contained in an $(n+1)$-element spanning set.
Prove that $\dim V = n$

Problem 10. Let B be a symmetric, non-degenerate, not positive definite bilinear form in an n-dimensional real vector space V. Prove that there exists a basis $v_1, \ldots, v_n$ in V such that $B(v_i, v_i) < 0$ for all i.

Problem 11. Let $I \subset \mathbb{R}[x]$ be the ideal generated by the polynomial $x^2 + 2x + 3$. Prove that the quotient ring $\mathbb{R}[x]/I$ is isomorphic to the field $\mathbb{C}$ of complex numbers.

Problem 12. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find α so that $K = \mathbb{Q}(\alpha)$, and compute the irreducible polynomial of α over $\mathbb{Q}$.