Final
MAT-022A

You have 100 minutes. You may only use a pencil (or pen) and the scrap paper that I provide. No calculators, notes or books. You must show your work to receive full credit.

1. Let $u$ and $v$ be orthogonal vectors such that $||u|| = 2$ and $||v|| = 1$. Find $||2u - 3v||$. (Your work must show your answer is true for any $u$ and $v$ that satisfy the above properties.) (10 points)

2. Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. (10 points)
3. Let \( N \) be the set of all \( 3 \times 3 \) nonsingular matrices with real entries. Prove or disprove that 
\( N \) is a real vector space under the operations of matrix addition and scalar multiplication. 
(10 points)

4. Let \( P_n \) denote the set of all polynomials of degree at most \( n \) with real coefficients (i.e. polynomials of the form \( a_0 + a_1t + a_2t^2 + \ldots + a_n t^n \) for \( a_0, a_1, \ldots, a_n \in \mathbb{R} \)) and let \( S = \{ t^2, 1+t, t^3+4 \} \). 
(15 points)

a) Is the set \( S \) linearly independent in \( P_3 \)?

b) Is the set \( S \) a basis for \( P_3 \)?

c) What is \( \text{dim}(P_n) \)? (no work necessary)
5. Let \( A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{bmatrix} \). (15 points)

a) Find a basis for the null space of \( A \).

b) Find a basis for the row space of \( A \).

c) Find a basis for the column space of \( A \).
d) What is the nullity of $A$?

e) What is the rank of $A$?

6. If $\text{nullity}(A) = 2$ for some $6 \times 8$ matrix $A$, what is the rank of $A$? (5 points)

7. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ are ordered bases for the set of all $2 \times 2$ diagonal matrices with real entries). Find the transition matrix $P_{S \leftarrow T}$ from the $T$-basis to the $S$-basis. (10 points)
8. Let \( P_{S\leftarrow T} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \) be a transition matrix from some ordered basis \( T \) to another ordered basis \( S \) of the same vector space. Find \( P_{T\leftarrow S} \), the transition matrix from the \( S \)-basis to the \( T \)-basis. (10 points)

9. Let \( A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \). Find an orthogonal matrix \( Q \) such that \( Q^T A Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). (15 points)
Extra Credit 1: Prove that for an $n \times n$ matrix $A$, the matrix transformation $f(x) = Ax$ is both one-to-one (i.e. $f(x) = f(y) \iff x = y$) and onto (i.e. for each $y$ in $\mathbb{C}^n$ there is an $x$ in $\mathbb{C}^n$ such that $f(x) = y$) if and only if $A$ is nonsingular. (10 points, test score can’t exceed 100 points)

Extra Credit 2: Let $A$ be a $3 \times 3$ matrix that rotates vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ around the $x$-axis by 45 degrees. Find a basis for the subspace of $\mathbb{R}^3$ spanned by the eigenvectors of $A$. (Hint: You don’t need to find $A$. Just ask yourself, ”What is special about eigenvectors?”) (10 points, test score can’t exceed 100 points)