MAT 25B Final Exam  (2013/12/12)

Honor Pledge: I pledge on my honor that I have not given or received any unauthorized assistance on this exam.

Name: 
Signature:

1. Show all your work. Jumping to right answers without minimum reasoning deserves no credit.
2. Answer each numbered problem on a separate answer sheet.
3. Open book. No electronic devices are allowed.
4. Read directions to each problem carefully.
5. Neatness and organization are also important.

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# 1. [40 pts] Determine whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true. (10 points each.)

(1) If $a < 1$ for all $a \in A$, then $\sup A < 1$.

(2) The union of an arbitrary collection of compact sets is compact.

(3) Let $a$ be a real number. Then, the set $\{a\}$ is closed.

(4) If $A$ and $B$ are connected, then $A \cup B$ is connected.
2. [20 pts] Let \((a_n)\) be a sequence such that \(a_n \geq 1\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} a_n = a\). Show that

\[
\lim_{n \to \infty} \sqrt{a_n} = \sqrt{a}.
\]
# 3. [40 pts] Determine whether the following series converges or diverges. (10 points each.)

(1) \( \sum_{n=1}^{\infty} \frac{2^n 3^n}{10^n} \) 
(2) \( \sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)} \) 
(3) \( \sum_{n=1}^{\infty} \frac{2n + 7}{n^2 + 5n} \) 
(4) \( \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} \)
# 4. [10 points each.] Let \((a_n)\) be a sequence defined as

\[ a_1 = 1, \quad a_{n+1} = \frac{4a_n + 5}{5} \quad \text{for } n \in \mathbb{N}. \]

(1) Show that the sequence \((a_n)\) satisfies \(a_n < 5\) for all \(n \in \mathbb{N}\).

(2) Show that the sequence \((a_n)\) is increasing.

(3) Show that the sequence \((a_n)\) converges. Compute \(\lim_{n \to \infty} a_n\).
# 5. Let $A$ and $B$ be subsets in $\mathbb{R}$.

(1) [10 pts] Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(2) [20 pts] Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. 

# 6. [20 pts] Let \((a_n)\) be a sequence such that \(\lim_{n \to \infty} a_n = a\). Show that the following set is compact

\[ K = \{a_n : n \in \mathbb{N}\} \cup \{a\}. \]
# 7. [20 pts] Let $A$ and $B$ be connected sets in $\mathbb{R}$ such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is connected.