Spring 2012: PhD Algebra Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1. Let $A$ be a real $n \times n$ upper triangular matrix so that $A$ commutes with its transpose $A^T$. Show that $A$ is diagonal.

Problem 2. Suppose that $G$ is a group which contains no index 2 subgroups. Show that every index 3 subgroup in $G$ is normal.

Problem 3. Let $F$ be a field and $F^\times$ be the multiplicative group of nonzero elements of $F$. Show that every finite subgroup of $F^\times$ is cyclic.

Problem 4. Prove that $\mathbb{R}[X]/(X^2-1)\mathbb{R}[X] \cong \mathbb{R} \oplus \mathbb{R}$, but $\mathbb{R}[X]/(X^2-1)^2\mathbb{R}[X] \not\cong \mathbb{R} \oplus \mathbb{R}$.

Problem 5. Show that 9 and $6 + 3\sqrt{-5}$ do not have a greatest common divisor in $\mathbb{Z}[\sqrt{-5}]$.

Problem 6. Let $F$ be a field, $X$ an indeterminate, and let $F[[X]]$ denote the ring of formal power series with coefficients in $F$, where multiplication is defined as it is for polynomials. Prove that an element $s = a_0 + a_1 X + \cdots \in F[[X]]$ is a unit in $F[[X]]$ if and only if $a_0 \neq 0$. Show that every ideal of $F[[X]]$ is of the form $X^n F[[X]]$ for some $n \geq 0$. 
