Fall 2013: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Find $\inf \int_0^1 |f(x) - x|^2 dx$ where the infimum is taken over all $f \in L^2([0,1])$ such that $\int_0^1 f(x)(x^2 - 1)dx = 1$.

Problem 2: Let $L^2([0,1])$ denote the Hilbert space of complex valued square integrable functions on $[0,1]$ with the usual inner product

$$(f, g) = \int_0^1 f(x)\overline{g(x)}dx.$$

Define $T : L^2([0,1]) \to L^2([0,1])$ by

$$(Tf)(x) = \int_0^x f(t)dt, \text{ for } x \in [0,1].$$

(a) Show that T is bounded.

(b) Show that T has no eigenvalues.

(c) Find $\lim_{n\to\infty} ||T^n||$.

Problem 3: For $\delta > 0$ small, let $u \in L^{\frac{3}{2}+\delta}(\mathbb{R}^3) \cap L^{\frac{3}{2}-\delta}(\mathbb{R}^3)$. Prove that $v = u * \frac{1}{|x|} \in L^\infty(\mathbb{R}^3)$ and provide a bound for $\|v\|_{L^\infty(\mathbb{R}^3)}$ which depends only on $\|u\|_{L^{\frac{3}{2}+\delta}(\mathbb{R}^3)}$.

Problem 4: Let H be a separable infinite dimensional Hilbert space and suppose that $e_1, e_2, \ldots$ is an orthonormal system in H. Let $f_1, f_2, \ldots$ be another orthonormal system which is complete (i.e. the closure of the span of $\{f_i\}_i$ is all of H.) Prove that if $\sum_{n=1}^\infty ||e_n - f_n||^2 < 1$ then $\{e_i\}_i$ is also a complete orthonormal system.

Problem 5: Suppose A is a compact operator on an infinite dimensional Hilbert space $\mathcal{H}$. Show that A does not have a bounded inverse operator.

Problem 6: Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwarz space. Show that $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for any $1 \le p \le \infty$.