Ordinary Differential Equations

Math 22B

Final Exam

NAME..

SIGNATURE..

I.D. NUMBER..

No books, notes, or calculators. Show all your work

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. [20 pts.] (a) Find the solution of the initial value problem

$$ty' = \frac{1}{y+1}, \qquad\qquad y(1) = 0.$$

(b) For what t-interval is the solution defined?

2. [20 pts.] Suppose that a is a constant, and consider the initial value problem

$$y' - y = e^{at}, \qquad y(0) = 0.$$

(a) Find the solution if $a \neq 1$.

(b) Find the solution if $a = 1$.

(c) Show that the solution in (b) is the limit of the solution in (a) as $a \to 1$. (Hint: use l'Hospital's rule.)

3. [20 pts.] Find the general solutions of the following ODEs.

(a) $y'' - 4y' + 5y = 0.$

(b) $y'' + 3y' - 4y = 0.$

4. [20 pts.] Find particular solutions of the following ODEs.

(a) $y'' - y' + 3y = \sin t$.

(b) $y'' + 2y' - 3y = e^t$.

5. [20 pts.] Suppose that the coefficient functions $p(t)$, $q(t)$ are continuous in the interval $0 < t < \pi$, and the functions $y_1(t) = t$, $y_2(t) = \sin t$ are solutions of the ODE

$$y'' + p(t)y' + q(t)y = 0 \qquad\qquad 0 < t < \pi.$$

(a) Compute the Wronskian of y_1, y_2. Are they linearly independent on the interval $0 < t < \pi$? Is the pair $\{y_1, y_2\}$ a fundamental set of solutions for the ODE? Could $p(t)$, $q(t)$ be continuous on $-\pi < t < \pi$? Explain your answers.

(b) Find the solution $y(t)$ of the initial value problem for the ODE with initial conditions

$$y\left(\frac{\pi}{2}\right) = 0, \qquad y'\left(\frac{\pi}{2}\right) = 2.$$

6. [20 pts.] The displacement $y(t)$ of an undamped oscillator of mass $m > 0$ on a spring with spring constant $k > 0$, and initial displacement $a \neq 0$ and initial velocity 0 satisfies

$$my'' + ky = 0, \qquad y(0) = a, \quad y'(0) = 0.$$

(a) Solve this initial value problem.

(b) Show that the solution is periodic with period T, meaning that $y(t+T) = y(t)$, and express T in terms of m and k.

(c) For what times t does the oscillator pass through equilibrium, meaning that $y(t) = 0$?

7. [20 pts.] (a) Find the general solution for $\vec{x}(t)$ of the following 2×2 system:

$$\vec{x}' = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \vec{x}.$$

(b) Classify the equilibrium $\vec{x} = 0$. Is it stable or unstable?

8. [20 pts.] Suppose that a 2×2 matrix A has the following eigenvalues and eigenvectors:

$$r_1 = 2, \quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = 1, \quad \vec{\xi}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Sketch the trajectories of the system $\vec{x}' = A\vec{x}$, where $\vec{x} = (x_1, x_2)^T$, in the phase plane. Classify the equilibrium $\vec{x} = 0$. Is it stable or unstable?

(b) Sketch the graphs of $x_1(t)$ and $x_2(t)$ versus t for the solution that satisfies the initial condition $x_1(0) = 2$, $x_2(0) = 0$.

9. [20 pts.] (a) Use the definition of the matrix exponential

$$e^{tA} = I + tA + \frac{1}{2!}t^2 A^2 + \ldots + \frac{1}{n!}t^n A^n + \ldots,$$

to compute e^{tA} for the following 2×2 matrix:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) Use your result from (a) to find the solution $\vec{x}(t) = (x_1(t), x_2(t))^T$ of the initial value problem

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Write out the solutions for the components $x_1(t)$, $x_2(t)$ explicitly.

10. [20 pts.] Suppose that $-1 < a < 1$ is a constant parameter, and $y(t)$ satisfies the ODE

$$y' = (a - y^2)(y - 2).$$

(a) Find the equilibria, sketch the phase line, and determine the stability of the equilibria in each of the following cases: (i) $-1 < a < 0$; (ii) $a = 0$; (iii) $0 < a < 1$.

(b) Suppose that $y(t)$ is the solution of the ODE that satisfies the initial condition $y(0) = 0$. What is the behavior of $y(t)$ as $t \to +\infty$ in each of the cases (i), (ii), (iii).