Winter 2007: Applied Math Preliminary Exam
Part I: Analysis

Instructions:

1. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1. Let $C([0, 1])$ be the Banach space of continuous real-valued functions on $[0, 1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0, 1]) \to C([0, 1])$ be a bounded linear operator. Suppose that $\|S(p)\| \leq 2$ for all polynomials $p$. Show that $S$ is the zero operator.

Problem 2. For $p \geq 1$, let $l^p(\mathbb{N})$ be the set of sequences $(x_n)$ such that

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

(a) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \subseteq l^q(\mathbb{N})$.

(b) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \not= l^q(\mathbb{N})$.

Problem 3. Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} f(x, y);$$

in particular, both limits exist. Does it follow that

$$\lim_{(x,y) \to (0,0)} f(x,y)$$

exists?

Problem 4. Let $X$ be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < C d(x, y)$ for all $x \neq y$. The function $f$ is said to be a weak contraction if $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, without the constant $C$. The contraction mapping theorem says that if $f$ is a contraction, then it has a fixed point. Show that the theorem also holds when $f$ is a weak contraction and $X$ is compact.

Problem 5. Construct the Green’s function for the Dirichlet boundary-value problem

$$-u'' + 4u = f, \quad u(0) = u(2) = 0.$$

Problem 6. Let $U$ be a unitary operator on a Hilbert space. Prove that the spectrum of $U$ lies on the unit circle.
1. Show that the system

\[
\begin{align*}
\dot{x} &= -x + y^3 - y^4 \\
\dot{y} &= -2x - y + 2xy
\end{align*}
\]

has no periodic solutions. What is the asymptotic behavior, as \( t \to \infty \), of the trajectory starting at \((\pi, -e^2)\)? (Hint: Choose \( a, m \) and \( n \) such that \( V = x^m + ay^n \) is a Liapunov function.)

2. Consider the system

\[\ddot{x} = -x^2 + (r - 2)x + r - 1,\]

where \( r \) is a parameter.

(a) Show that there is a bifurcation at \( r = r_c \) for some \( r_c \). Find the value of \( r_c \). What kind of bifurcation is it?

(b) Classify the fixed points of this nonlinear system. Give your reasons.