Characterizing Singular Polynomials for Dunkl Operators

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1 Introduction

Let $f$ be a polynomial in $n$ variables with coefficients in $\mathbb{R}$. Define $s_{ij}(f)$ as an operator which transposes $x_i$ and $x_j$ in $f$, e.g. if $f = ax_1 + bx_2 + cx_3$, then $s_{1,3}(f) = ax_3 + bx_2 + cx_1$.

**Definition 1.1.** Now define the Dunkl Operator as:

$$D_i(f) = \frac{\partial f}{\partial x_i} - c \sum_{i \neq j} \frac{f - s_{ij}}{x_i - x_j},$$

where $i, j$ span over all possible $n$ and $c$ is some constant.

Dunkl Operators have many applications in representation theory of rational Cherednik algebras, algebraic geometry, physics, and many other fields. In particular, Dunkl Operators are used to study the Calogero-Moser system; a discrete system of $n$ particles. Specifically, we have:

$$\sum D_i^2 = H = \sum \frac{\partial}{\partial x_i^2} + k \sum \frac{1}{(x_i - x_j)^2},$$

where $k$ is some constant dependent on $c$ and $H$ is the quantum Calogero-Moser Hamiltonian. [3] [4]

**Definition 1.2.** A singular polynomial for the Dunkl Operator is a polynomial $f \in \mathbb{R}[x_1 \ldots x_n]$ such that $D_i(f) = 0$ for all $i \in \{1, \ldots, n\}$. [1]

Our objective is to characterize these singular polynomials and to determine their dependence on the constant $c$.

**Main Results.** The following are the main results contained in this paper.

1. If $f$ is a degree 1 polynomial, then $f$ is singular if and only if its coefficients sum to 0 and $c = \frac{1}{n}$.
2. Vandermonde determinants are singular for $c = \frac{1}{2}$.
3. Symmetric polynomials of degree greater than 0 are never singular.
4. There always exists a quadratic singular polynomial in $n$ variables for $c = \frac{2}{n}$.
5. Any Garnir polynomial corresponding to a Young diagram $\lambda$ such that $\lambda$ corresponds to the partition $(b - 1, n - b + 1)$ where $b \geq \frac{n}{2}$ is singular for $c = \frac{1}{b}$.
6. Conjecture: Let $\lambda$ be a Young diagram of size $n$ such that $n = (b - 1)q + r$, for some $b > 0, q > 0, r < b - 1$, where $\lambda$ has $q$ rows of $(b - 1)$ boxes and one row of $r$ boxes. Then, any Garnir polynomial associated with $\lambda$ is singular if and only if $c = \frac{1}{b}$. Furthermore, any Young diagram not adhering to these requirements has no associated Garnir polynomials which are ever singular.
Proposition 1.3. [1] The following are important properties of Dunkl Operators:

1. \( D_i D_j = D_j D_i \)
2. \( D_i \) is linear.

Theorem 1.4. [1, 2] If singular polynomials of positive degree exist, then \( c = \frac{a}{b} \), where \( a, b \) are integers and \( b \leq n \).

Proposition 1.5. The space of singular polynomials forms a representation of \( S_n \).

2 Degree 1 Polynomials

Theorem 2.1. Let \( f = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \) be some degree 1 polynomial in \( n \) variables. Then \( f \) is singular if and only if \( a_1 + a_2 + \ldots + a_n = 0 \) and \( c = \frac{1}{n} \).

Proof. It is easily seen that \( D_i(x_i) = 1 - c(n-1) \) and \( D_i(x_j) = c \) for \( i \neq j \).

Now we have a system of equations involving each \( D_i \). Take this system of linear equations and construct the \( n \) by \( n \) matrix \( A \) below:

\[
A = \begin{pmatrix}
1 - c(n-1) & c & \cdots & c \\
c & 1 - c(n-1) & c & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
c & \cdots & 1 - c(n-1)
\end{pmatrix}
\]

Let \( a = 1 - c(n-1) \). Then

\[
\det(A) = \prod_{i=0}^{i=n-1} \frac{(a + ic)(a - c)}{a + (i-1)c}.
\]

This formula was found by row reducing the matrix above to an upper triangular matrix and then multiplying along the diagonal. We use induction to prove that this formula holds.

Suppose \( B \) is a \( k \times k \) matrix with \( 1 - c(n-1) \) on the main diagonal and \( c \) everywhere else. It is easy enough to check that \( \det(B) = \prod_{i=0}^{i=k-1} \frac{(a + ic)(a - c)}{a + (i-1)c} \) for \( k = 2 \).

Let \( k = 2 \) and let:

\[
B = \begin{pmatrix}
a & c \\
c & a
\end{pmatrix}
\]

where \( a \) is defined as above.

Then, after row reducing \( B \) we have the matrix:

\[
B' = \begin{pmatrix}
a & c \\
0 & a - c^2
\end{pmatrix}
\]
and \( \det(B^*) = \det(B) = a \cdot \frac{(a+c)(a-c)}{a} \) as desired.

For the inductive step, suppose that for some \( k-1 \) we have that \( \det(B) \) is as desired. Now for \( k \) we can construct the \( k \times k \) matrix:

\[
B' = \begin{pmatrix}
1 - c(n-1) & c & \ldots & c \\
c & 1 - c(n-1) & c & \ldots \\
0 & \vdots & \ddots & \vdots \\
c & \ldots & 1 - c(n-1)
\end{pmatrix}
\]

where the upper left \((k-1) \times (k-1)\) block matrix is \(B\).

Now we wish to row reduce \( B' \). To do this, note that the row reduction process for the first \( k-1 \) rows of \( B' \) will be almost exactly the same as the row reduction process for \( B \). Thus, we need only focus on the reduction of the \( k \)th row of \( B' \). At the end of the \((k-1)\)st step we will have the matrix:

\[
\begin{pmatrix}
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \frac{(a+(k-1)c)(a-c)}{a+(k-2)c} & \frac{-c^2 + ac}{a+(k-2)c} \\
0 & \ldots & \frac{-c^2 + ac}{a+(k-2)c} & \frac{(a+(k-1)c)(a-c)}{a+(k-2)c}
\end{pmatrix}
\]

To show that the \((k-1,k)\) and \((k,k-1)\) positions are both equal to \( \frac{-c^2 + ac}{a+(k-2)c} \), we use induction. In general, we wish to show that after the \( i+1 \) row has been reduced, we have that the non-zero and non-diagonal elements in that row are equal to \( \frac{-c^2 + ac}{a+(k-2)c} \). This can be easily shown to hold for \( i+1 = 2,3 \) so we can move on to the induction step.

Assume we have that this holds for some row \( i \). Then the \( i \) and \( i+1 \) rows both have \( \frac{-c^2 + ac}{a+(i-1)c} \) in their non-zero, non-diagonal positions. Observe that we can further row reduce by multiplying row \( i \) by \( \frac{a}{a+ic} \) and adding it to row \( i+1 \). This gives:

\[
\frac{-c^2 + ac}{a+(i-1)c} \cdot \frac{-c}{a+ic} + \frac{-c^2 + ac}{a+(i-1)c} = \frac{(-c^2 + ac)(a+ic-c)}{(a+(i-1)c)(a+ic)} = \frac{-c^2 + ac}{a+ic}
\]

which is as we wished to show. Now, if we let \( i = k-2 \) (for the \( k-1 \) row) we get the entries above.

For the final part of the row reduction process, multiply the top row by \( \frac{-c}{a+(k-1)c} \) and add it to the bottom row to form a new bottom row. The row reduced form, \( B^* \), of \( B' \) is:

\[
B^* = \begin{pmatrix}
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \frac{(a+(k-1)c)(a-c)}{a+(k-2)c} & \frac{-c^2 + ac}{a+(k-2)c} \\
0 & \ldots & 0 & \frac{(a+k_c)(a-c)}{a+(k-1)c}
\end{pmatrix}
\]

which gives us \( \det(B) = \prod_{i=0}^{k-1} \frac{(a+(i)c)(a-c)}{a+(i-1)c} \).

Now, if we let \( k = n \), we get \( \det(A) = \prod_{i=0}^{n-1} \frac{(a+(i)c)(a-c)}{a+(i-1)c} \) as desired.
For each $i$, the roots of the numerator in the product are $c = a, c = -\frac{a}{i}$. The $(a + ic)$ in the numerator of the $i$th term will cancel with the $(a + ic)$ in the denominator of the $i + 1$ term. Additionally, the $(a + ic)$ in the numerator of the $i = n - 1$ term becomes 1 when evaluated at $a = 1 - c(n - 1)$; thus, the $-\frac{a}{i}$ root can be discarded. Only the $c = a$ root is left. Substituting $a$ for $1 - c(n - 1)$, gives $c = \frac{1}{n}$. Substituting $c = \frac{1}{n}$ into $A$ gives a matrix where all entries are $\frac{1}{n}$.

3 The Vandermonde Determinant and Symmetric Polynomials

Definition 3.1. Let $W = \prod_{i<j} (x_i - x_j)$ where the product is taken over $n$ variables. A polynomial of this form is called a Vandermonde Determinant.

Proposition 3.2. Let $W$ be the Vandermonde Determinant. Then $s_{ij}(W) = -W$.

Proof. Let $W$ be the Vandermonde Determinant in $n$ variables. First consider the action of $s_{ij}$ on $W$. Note that $s_{ij}(W) = \prod_{i<j} s_{ij}[(x_i - x_j)]$. Without loss of generality, suppose $i < j$. We can ignore all factors $(x_k - x_i)$ such that $k, t \neq i, j$ since $s_{ij}$ will have no effect on them. We now have the following four cases to consider.

\[
(x_i - x_k) \\
(x_k - x_i) \\
(x_j - x_k) \\
(x_k - x_j) \\
(x_i - x_j)
\]

Where $k \neq i, j$.

Note that, given any $k$, $i$ and $j$ will be in some factor of $W$ together with $k$. Thus, when we apply $s_{ij}$ to $W$ we will get back all the factors of $W$ multiplied by $-1^\mu$ for some $\mu$. Thus, we only need to know the parity of $\mu$ to determine if $s_{ij}(W) = W$ or $-W$.

First consider the $(x_i - x_k)$ factors. If $k > j$ then $s_{ij}(x_i - x_k) = (x_j - x_k)$. If $k < j$ then

\[s_{ij}(x_i - x_k) = (x_j - x_k) = -(x_k - x_j)\]

and so we gain a factor of $-1$. There are $j - i - 1$ of these $(x_i - x_k)$ factors where $k < j$.

Now consider the $(x_k - x_i)$ and the $(x_j - x_k)$ factors. If $k < i$ then $k < j$ and if $k > j$ then $k > i$. Thus,

\[s_{ij}(x_k - x_i) = (x_k - x_j)\]

and we pick up no factors of $-1$.
Next we look at the \((x_k - x_j)\) factors. If \(i > k\) then \(s_{ij}\) gives back the original factor. If \(i < k\), then
\[
s_{ij}(x_k - x_j) = (x_k - x_i) = -(x_i - x_k)
\]
and so we gain a factor of \(-1\). There are \(j - i - 1\) of these \((x_k - x_j)\) factors where \(i < k\).

Finally, we have the solitary \((x_i - x_j)\) factors. Clearly we have, \(s_{ij}(x_i - x_j) = -(x_i - x_j)\) and so we gain a single factor of \(-1\).

We see that \(\mu = 1 + 2(j - i - 1)\), which is odd. Therefore, \(s_{ij}(W) = -W\).

**Corollary 3.3.** Let \(W\) be the Vandermonde Determinant and let \(s \in S_n\). Then \(s(W) = sgn(s)W\).

**Proof.** By the Proposition above, \(s_{ij}(W) = -W\). Note that \(sgn(s_{ij}) = -1\). Since any \(s \in S_n\) can be decomposed into a product of transpositions we have,
\[
s(W) = (-1)^\mu W
\]
where \(\mu\) is number of transposition in the decomposition of \(s\). In other words, \((-1)^\mu = sgn(s)\). \(\Box\)

**Corollary 3.4.** The Vandermonde Determinant \(W\) spans a one dimensional representation of \(S_n\). This representation is isomorphic to the sign representation.

**Theorem 3.5.** For any \(n\), the Vandermonde Determinant \(W\) is singular if and only if \(c = \frac{1}{2}\).

**Proof.**
\[
A = -c \sum_{i \neq j} \frac{W - s_{ij}(W)}{x_i - x_j} = -c \sum_i \frac{2W}{x_i - x_j} = -2c \sum_{i \neq j} \frac{W}{x_i - x_j}
\]
(1)

Now let,
\[
\Gamma_n = \prod_{k<p} (x_k - x_p)
\]
where \(k, p \neq i\). Then,
\[
A = -2c\Gamma_n \sum_{i \neq j} \left[ (-1)^\mu \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \right]
\]
where \(\mu = 0\) if \(i < j\), \(1\) if \(i > j\).

We want to show that
\[
\frac{\partial W}{\partial x_i} = A \frac{1}{2c} \text{ for all } i,
\]
we do this by induction.
It is easy enough to show that (2) holds for \( n = 1, 2 \) and so we have base cases completed.

Now for the inductive step. Let \( W_n \) and \( W_{n+1} \) be the Vandermonde Determinants over \( n \) and \( n+1 \) variables respectively. Then,

\[
W_{n+1} = (x_1 - x_{n+1}) \cdots (x_n - x_{n+1}) W_n
\]

Thus, we have

\[
\frac{\partial W_{n+1}}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ (x_1 - x_{n+1}) \cdots (x_n - x_{n+1}) W_n \right]
\]

If \( i \neq n+1 \),

\[
\frac{\partial W_{n+1}}{\partial x_i} = (x_1 - x_{n+1}) \cdots (x_n - x_{n+1}) \frac{\partial W_n}{\partial x_i} + \frac{\partial}{\partial x_i} \left[ (x_1 - x_{n+1}) \cdots (x_n - x_{n+1}) \right] W_n
\]

\[
= (x_1 - x_{n+1}) \cdots (x_n - x_{n+1}) \Gamma_n \times
\]

\[
\sum_{i,j \in [n], i \neq j} (-1)^\mu \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i)
\]

\[
\frac{(x_1 - x_{n+1}) \cdots (x_n - x_{n+1})}{(x_i - x_{n+1})} W_n
\]

If \( i = n+1 \), then

\[
\frac{\partial W_{n+1}}{\partial x_i} = 0 + W_n \left[ \sum_{1 \leq j \leq n} \left( - \prod_{i=1, i \neq j}^n (x_i - x_{n+1}) \right) \right]
\]

\[
= \Gamma_{n+1} \sum_{i,j \in [n+1], i \neq j} (-1)^\mu \prod_{i<k,k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i)
\]

Thus, we have (2) and,

\[
D_i(W) = \Gamma_n \sum_{i,j \in [n+1], i \neq j} (-1)^\mu \prod_{i<k,k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \left( 1 - 2c \right)
\]

This implies that for any \( n \), \( W \) is singular if and only if \( c = \frac{1}{2} \). \( \Box \)

**Theorem 3.6.** Let \( g \) be a symmetric polynomial in \( n \) variables of degree at least 1. Then \( g \) is never singular.
Proof. Let \( g \) be a symmetric polynomials in \( n \) variables such that \( \deg(g) \geq 1 \). Then \( s_{ij}(g) = g \). Thus,
\[
\sum_{j \neq i} \frac{g - s_{ij}(g)}{x_i - x_j} = 0
\]
for all \( i \). Since \( g \) is not constant, we have \( \frac{\partial g}{\partial x_i} \neq 0 \). Therefore,
\[
D_i(g) = \frac{\partial g}{\partial x_i} \neq 0 \text{ for all } i.
\]

\[\square\]

4 Quadratic Polynomials

4.1 SAGE Calculations

Next we characterize quadratic singular polynomials for \( n = 3 \) and \( n = 4 \). Below are bases for the space of quadratic singular polynomials for certain values of \( c \) and \( n \). To find these bases, we use the following procedure:

1. Take the polynomial
\[
f = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + b_{1,2}x_1x_2 + b_{1,3}x_1x_3 + \ldots + b_{n-1,n}x_{n-1}x_n
\]
where the \( a_i, b_{i,j} \) are unknown coefficients.

2. Find \( D_i(f) \) for all \( i \).

3. Combine like terms in the \( D_i \) and consider the coefficient of each term. These coefficients will be a combination of the \( a_i \)'s.

4. We require that these coefficients equal 0 for each term to vanish (note that since like terms have already been collected, different terms cannot cancel with one another). Thus, we can create a system of linear equations, using the \( a_i \) as our unknowns and setting each of these coefficients equal to 0.

5. Solve this system of equation and use it to determine a basis for all singular quadratic polynomials in \( n \) variables for the given \( c \).

Proposition 4.1. Let \( f \) be a quadratic polynomial in 3 variables with \( c = \frac{2}{3} \). Then \( f \) is singular if and only if it is in the span of
\[
(x_2 - x_3)(-2x_1 + x_2 + x_3)
\]
\[
(x_1 - x_3)(x_1 + x_3 - 2x_2)
\]

Proposition 4.2. Let \( f \) be a quadratic polynomial in 4 variables with \( c = \frac{1}{2} \). Then \( f \) is singular if and only if it is in the span of
\[
(x_1 - x_4)^2 - (x_2 - x_3)^2
\]
\[
(x_1 - x_3)^2 - (x_2 - x_4)^2
\]
\[
(x_1 - x_2)^2 - (x_3 - x_4)^2
\]
Proposition 4.3. Let $f$ be a quadratic polynomial in 4 variables with $c = \frac{1}{3}$. Then $f$ is singular if it is in the span of

$$(x_1 - x_2)(x_3 - x_4)$$
$$(x_1 - x_3)(x_2 - x_4)$$

4.2 Decomposition of $P_{n,2}$

To further characterize quadratic singular polynomials, we investigate the decomposition of the space of quadratic polynomials into irreducible representations of $S_n$. Let $P_{n,2}$ be the space of quadratic polynomials in $n$ variables. We can decompose $P_{n,2}$ into irreducible representations of $S_n$ in the way shown in Figure 1.

![Diagram of decomposition of $P_{n,2}$](image)

Here $\dim V_1 = \dim V_3 = 1, \dim V_2 = \dim V_4 = n - 1, \dim V_5 = \frac{n(n-3)}{2}$ and $V_1, V_3$ are the trivial representations of $S_n$.

We can show that $V_1, V_2$ are irreducible by noting that $\text{span}(x_i^2)$ is isomorphic to $\mathbb{R}^n$. Under the permutation representation over $\mathbb{R}^n$, it is well known that $\mathbb{R}^n$ is the direct sum of two irreducible representation. One is the trivial representation, to which $V_1$ is clearly isomorphic, which leaves $V_2$ as the other irreducible representation. Also, $V_5$ is isomorphic to the irreducible representation of $S_n$ formed by the Garnir polynomials (discussed in section 5) and so $V_5$ is irreducible. Finally, $V_3 \oplus V_4 = \text{span}(\sum_{j \neq i, i \text{ fixed}} x_i x_j) \cong \mathbb{R}^n$. Since $V_3$ is isomorphic to the trivial representation (which is irreducible), we automatically have that $V_4$ is irreducible.

The Young diagrams below $V_2, V_4$, and $V_5$ correspond to these spaces in a
way explained in section 5. For the moment, consider the Young diagram:

\[
\lambda_{V_5} = \begin{array}{cccc}
\bullet & \bullet & & \\
& \bullet & & \\
& & & \\
& & & \\
\end{array}
\]

which represents \( V_5 \). As will be discussed in section 5, the number of SYT on \( \lambda_{V_5} \) equals the dimension of \( V_5 \). Thus, using the hook length formula, also discussed in section 5, we have:

\[
\dim V_5 = \frac{n!}{2 \cdot (n-1) \cdot (n-2) \cdot (n-4)!} = \frac{n(n-3)}{2}.
\]

Now we can refine our investigation into quadratic singular polynomials by using these \( V_i \). Notice that all elements of \( V_1 \) and \( V_3 \) are symmetric polynomials. We can conclude, by Theorem 3.6, that there exist no non-trivial singular polynomials in either of these spaces.

Next we look at \( V_5 \). Let \( S \) be the space of singular polynomials. As mentioned before, \( S \) is a representation of \( S_n \). Consider \( Q = S \cap V_5 \). The intersection of two representations of \( S_n \) is also be a representation of \( S_n \). Since \( V_n \) is irreducible, this implies that either \( Q = \{0\} \) or \( Q = V_5 \). Thus, if we show that any element of \( V_5 \) is singular, then we have that all elements in \( V_5 \) are singular. From this, we have the following.

**Theorem 4.4.** Let \( f \in V_5 \), then \( f \) is singular, for all \( n \), if and only if \( c = \frac{1}{(n-1)} \).

**Proof.** Note that \( f = (x_1 - x_2)(x_3 - x_4) \in V_5 \), this follows from Theorem 5.7. Thus, by the above argument, it suffices to show that \( f \) is singular. We have five cases to consider: \( D_1(f), D_2(f), D_3(f), D_4(f), D_j(f) \) where \( j \in [n] \) and \( j \neq 1,2,3,4 \). First:

\[
D_1(f) = (1 - 3c)(x_3 - x_4) - c(n-4)(x_3 - x_4) = (1 - c(n-1))(x_3 - x_4)
\]

\[\Rightarrow D_1(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}.\]

Since \( x_1 \) and \( x_3 \) behave in the same way, we also have:

\[
D_3(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}.
\]

Now:

\[
D_2(f) = (3c - 1)(x_3 - x_4) + c(n-4)(x_3 - x_4) = (c(n-1) - 1)(x_3 - x_4)
\]

\[\Rightarrow D_2(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}.\]

Similarly, we have that \( x_2 \) and \( x_4 \) behave in the same way, so:

\[
D_4(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}.
\]
Finally:

\[ D_j(f) = -(x_3 - x_4) + (x_3 - x_4) - (x_1 - x_2) + (x_1 - x_2) = 0. \]

Thus, we have that \( f \) is singular and therefore all elements in \( V_5 \) are singular. Furthermore, from Theorem 5.7 we have that \( V_5 = \text{span}\{ G_{T_{V_5}} \} \), where \( \{ G_{T_{V_5}} \} \) is the set of all Garnir polynomials corresponding to \( \lambda_{V_5} \) (Garnir polynomials will be discussed in the next section). Due to the shape of \( \lambda_{V_5} \), all elements in \( \{ G_{T_{V_5}} \} \) will behave the same way as \( f \) did when acted upon by the Dunkl Operators. Thus, each element in \( \{ G_{T_{V_5}} \} \) will be singular if and only if \( c = \frac{1}{n-1} \); and since any element of \( V_5 \) can be expressed as a linear combination of these polynomials, we have that any element of \( V_5 \) is also singular if and only if \( c = \frac{1}{n-1} \).

Finally, consider \( W = V_2 \oplus V_4 \). As with \( V_5 \) we look at \( H = S \cap W \). As before, \( H \) must be a representation of \( S_n \); however, \( W \) is not irreducible so we cannot come to the same conclusion we did with \( V_5 \). Instead, we see that \( H = AV_2 + BV_4 \) for some \( A, B \in \mathbb{C} \). This leads us to the following theorem.

**Theorem 4.5.** Let,

\[ f_q = \frac{1}{4} \frac{(n-1)(n-2)}{n^2} x_q^2 - \frac{n-2}{4} \sum_{i \neq q} x_i^2 - \frac{n-2}{2} \sum_{i \neq q} x_q x_i + \sum_{i,j \neq q} x_i x_j \]

where \( f \in \mathbb{R}[x_1 \ldots x_n] \); then, when \( c = \frac{2}{n} \), \( f \) is singular for all \( n \).

Before we begin the proof, note that the first two terms in \( f_q \) are elements of \( V_2 \) and the last two terms of \( f_q \) are elements of \( V_4 \).

**Proof.** Since \( q \) is arbitrary, we can prove the theorem holds for \( q = 1 \) and we will have proven it true for all \( q \). There are two cases to consider. The first is the action of \( D_1 \) on \( f \) and the second is the action of \( D_i \), where \( t \neq 1 \), on \( f \).

To find \( D_1(f) \), we apply the Dunkl Operator to each term of \( f \) separately. This gives,

\[
D_1(f) = \frac{2(n-1)(n-2)}{4n} x_1 - \frac{2(n-1)(n-2)}{4n} \sum_{i \neq 1} x_i - \frac{n-2}{2n} \sum_{i \neq 1} (x_1 + x_i) \\
+ \frac{(n-2)(n-4)}{2n} \sum_{i \neq 1} x_i + \frac{2}{n} \sum_{i,j \neq 1} (x_i + x_j)
\]

Summing the coefficients for the \( x_1 \) terms we have,

\[
\frac{2(n-1)(n-1)}{4n} - \frac{(n-2)(n-1)}{2n} = 0
\]
Similarly, for the $x_i$ terms we have,

\[-\frac{2(n-1)(n-2)}{4n} - \frac{(n-2)}{2n} + \frac{(n-2)(n-4)}{2n} + \frac{2(n-2)}{n} = 0\]

Now, for the $D_t$ case,

\[D_t(f) = \frac{(n-1)(n-2)}{2n} (x_1 + x_t)\]

\[+ \left[ -\frac{(n-2)}{2n} \sum_{i \neq t, 1} (x_i + x_t) - \frac{(n-2)}{2n} x_t + \frac{(n-2)}{2n} \sum_{i \neq t} x_i \right]\]

\[+ \left[ -\frac{(n-2)}{n} \sum_{i \neq t, 1} (x_1 + x_i) + \frac{(n-2)(n-4)}{2n} x_1 \right]\]

\[+ \left[ \frac{2}{n} \sum_{i,j \neq t, 1} (x_i + x_j) - \frac{n-4}{n} \sum_{i \neq 1, t} x_i \right]\]

In this case we must check the coefficients for $x_1, x_t$, and $x_i$.

For $x_1$ we have,

\[\frac{(n-1)(n-2)}{2n} + \frac{(-2n+4)(n-2)}{2n} + \frac{(n-2)(n-4)}{2n} + \frac{(n-2)}{2n} = 0\]

For $x_t$,

\[\frac{(n-1)(n-2)}{2n} - \frac{(n-2)(n-2)}{2n} - \frac{(n-2)}{2n} = 0\]

For $x_i$,

\[-\frac{(n-1)}{2n} + \frac{(n-2)}{2n} - \frac{(n-2)}{n} + \frac{2(n-3)}{n} - \frac{n-4}{n} = 0\]

And so we have the desired result. \(\square\)

**Proposition 4.6.** $F = \sum_{q=1}^{n} f_q = 0$.

**Proof.** Consider $x_j^2$ where $j$ is fixed. Then for $f_j$ we have $\frac{(n-1)(n-2)}{4}$ as the coefficient of this term. Now consider all other $f_q$, there are $(n-1)$ such $f_q$ and for each the coefficient of $x_j$ is $-\frac{(n-2)}{4}$. Thus we have:

\[\frac{(n-1)(n-2)}{4} - \frac{(n-1)(n-2)}{4} = 0\]

and the coefficient of $x_j^2$ in the sum is 0. Since $j$ is arbitrary, $F$ has no terms of the form $x_i^2$. 

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Similarly, take $x_{j,p}$ such that $j, p$ are fixed. Then for $f_j$ there are $-\frac{(n-2)}{2}$ such terms and for $f_p$ there are also $-\frac{(n-2)}{2}$ such terms. Now, looking at all other $f_q$, each has only one copy of $x_{j,p}$ and there are $(n - 2)$ remaining $f_q$. So:

$$-\frac{(n-2)}{2} - \frac{(n-2)}{2} + (n - 2) = 0.$$ 

Again, $i$ and $j$ are arbitrary so no such terms appear in $F$. Since these are all possible terms in $F$, we have $F = 0$. 

5 Young Diagrams and Garnir Polynomials

5.1 Young Diagrams

In this section, we recall some facts from combinatorics and representation theory [5]. In particular, standard Young tableaux will be discussed. As a point of clarity, note that we will use the convention of writing Young diagrams such that the longest row of boxes will be at the bottom of the diagram and the shortest row will be at the top. For instance, if we had the partition $(5, 4, 2)$, the corresponding Young diagram would be:

$$\lambda = \begin{array}{ccc}
\hline
& & \\
& & \\
& & \\
& & \\
& & \\
\hline
\end{array}$$

**Definition 5.1.** Let $\lambda$ be some Young diagram. A Standard Young Tableau (SYT), denoted $T_\lambda$, is a filling of the boxes of $\lambda$ with the numbers $1, \ldots, n$, where $n$ is the number of boxes in $\lambda$. These numbers must be placed in $\lambda$ such that, when moving from the bottom of any column to the top, the numbers increase and similarly, when moving from left to right along a row, the numbers also increase.

For example, if $\lambda$ is the Young diagram corresponding to the partition, $(4, 3, 2)$, we could have the SYT:

$$T_\lambda = \begin{array}{ccc}
4 & 6 & \\
2 & 5 & 9 \\
1 & 3 & 7 & 8
\end{array}$$

However, the following filling:

$$A = \begin{array}{ccc}
2 & 6 & \\
4 & 5 & 9 \\
1 & 3 & 7 & 8
\end{array}$$

would not be a SYT because in the first column, numbers do not increase from bottom to top, we move from 4 in the box of the middle row to 2 in the box of the top row.
**Definition 5.2.** Let $\lambda$ be some Young diagram. Let $a$ be some box in $\lambda$. Let $u_i$ be the number of boxes, in the same column as $a$, above $a$ and let $r$ be the number of boxes, in the same row of $a$, to the right of $a$. Then $h_a = u + r + 1$ is the hook length of $a$.

The following is a well known fact in representation theory.

**Hook Length Formula 5.3.** Let $\lambda$ be a Young diagram for some partition of $n$. Let $f_\lambda$ be the number of standard Young tableaux that can be formed from $\lambda$, that is, the number of ways the boxes of $\lambda$ can be filled with the numbers $1, \ldots, n$ such that the rules for SYT are obeyed. Then:

$$f_\lambda = \frac{n!}{\prod_{a \in \lambda} h_a}.$$ 

This is called the hook length formula.

For example, consider:

$$Y = \begin{array}{ccc} 2 & 1 \\ 5 & 4 & 2 \\ 1 \end{array}$$

The tableau $Y$ is not a standard Young tableau. Rather, $Y$ is a Young diagram where the boxes of $Y$ have been filled with their respective hook length. The box in the bottom left corner has 3 boxes to the right of it and 1 above it, thus its hook length is 5. Then, from the hook length formula, we have that the number of possible SYT on this shape is:

$$f_\lambda = \frac{6!}{2 \cdot 1 \cdot 5 \cdot 4 \cdot 2 \cdot 1} = 9$$

**Theorem 5.4.** The irreducible representations of the symmetric group $S_n$ are in bijection with the Young diagrams that give partitions of $n$.

Recall the figures below the $V_i$ in the earlier decomposition of the space of quadratic polynomials in $n$ variables. Since each $V_i$ is an irreducible representation of $S_n$, these figures were the Young diagrams corresponding to those particular irreducible representations.

**Theorem 5.5.** Let $V$ be some irreducible representation of $S_n$ and let $\lambda$ be the Young diagram associated with $V$. Then:

$$\dim V = f_\lambda$$

where the SYT are taken with respect to shape $\lambda$ and $f_\lambda$ is found using the hook length formula given above.
5.2 Garnir Polynomials

Definition 5.6. Let $T_\lambda$ be a standard Young tableaux for some Young diagram $\lambda$. Denote each column of $T_\lambda$ by $a^i$, so column one (the column farthest to the left) is $a^1$, column two is $a^2$, and so on. Now denote the number in a box of $T_\lambda$ by $a^i_j$ where $j$ is the box’s position (counting from bottom to top) in column $i$. Thus, the filling in the lower left box will be denoted by $a^1_1$, the filling of the box above it by $a^1_2$, the filling of the box to its immediate right by $a^2_1$, and so on. Associated with $T_\lambda$ is the Garnir polynomial $G_{T_\lambda}$ where:

$$G_{T_\lambda} = \prod_{i=1}^{q} \left[ \prod_{j=1}^{\lambda_i} \left( \prod_{k>j}(x_{a^i_j} - x_{a^i_k}) \right) \right]$$

where $q$ is the number of columns in $T_\lambda$, $\lambda_i$ is the number of boxes in column $i$, and $k \leq \lambda_i$ [6].

For example, let:

$$T_\lambda = \begin{array}{cccc}
3 & 6 \\
2 & 5 & 8 \\
1 & 4 & 7 & 9
\end{array}$$

Then:

$$G_{T_\lambda} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5)(x_4 - x_6)(x_5 - x_6)(x_7 - x_8)$$

Notice that $x_9$ is not part of any factor since there are no boxes above the box containing the number 9.

Now let:

$$T_\lambda = \begin{array}{c}
4 \\
3 \\
2
\end{array}$$

Then:

$$G_{T_\lambda} = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

Notice that in this case, $G_{T_\lambda}$ is the Vandermonde determinant for $n = 4$.

Finally:

$$T_\lambda = \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5
\end{array}$$

Then:

$$G_{T_\lambda} = 1$$

$G_{T_\lambda}$ is a constant because $T_\lambda$ consists of only one row, so no factors with any $x_i$ can be formed.
Theorem 5.7. [6] Let $\lambda$ be some Young diagram. Let $\{G_{\lambda}\}$ be the set of all possible Garnir polynomials on $\lambda$. Then $\{G_{\lambda}\}$ is an irreducible representation of $S_n$ and $\{G_{\lambda}\}$ is isomorphic to the irreducible representation of $S_n$ corresponding to $\lambda$.

Recall $V_5$ in the decomposition of $P_{n,2}$. The Young diagram below $V_5$ in Figure 1 corresponds to $V_5$ itself. By the above theorem, this Young diagram, and by implication $V_5$, is isomorphic to the irreducible representation of $S_n$ formed from the Garnir polynomials of the Young diagram. Thus, $V_5$, as stated earlier, is irreducible.

Next we consider a specific kind of Young diagram. Let, $\lambda$ be the Young diagram corresponding to the partition $(b - 1, n - b + 1)$, where $b \geq \frac{n}{2}$. Let $T_\lambda$ be a standard Young Tableau over $\lambda$ such that, if $k$ is the filling of some box, then the number occupying the box directly above it in the same column is $k + 1$. Now let $G_{T_\lambda}$ be the Garnir polynomial corresponding to $T_\lambda$. Thus, if $n = 5, b = 4$, then:

$$T_\lambda = \begin{array}{ccc} 2 & 4 \\ 1 & 3 & 5 \end{array}$$

and

$$G_{T_\lambda} = (x_1 - x_2)(x_3 - x_4)$$

An interesting interpretation of $T_\lambda$ is that it can be constructed from any $b$ satisfying:

$$n = (b - 1)1 + (n - b + 1)$$

where $(b - 1)$ gives the number of boxes in the bottom row and $(n - b + 1)$ gives the number of boxes in the top row. In other words, it is a way of depicting division with remained of $n$ by $b - 1$.

From this construction, we have the following theorem.

**Theorem 5.8.** Let,

$$G_{T_\lambda} = (x_1 - x_2)(x_3 - x_4) \ldots (x_{2(n-b+1)-1} - x_{2(n-b+1)})$$

where $G_{T_\lambda}$ is the Garnir polynomial described above and $b \geq \frac{n}{2}$. Then $G_{T_\lambda}$ is singular if and only if $c = \frac{1}{b}$.

**Proof.** There are three cases to consider, $D_t(G_{T_\lambda})$ when

1. $t$ is even, and present in $G_{T_\lambda}$
2. $t$ is odd, and present in $G_{T_\lambda}$
3. $t$ is not in $G_{T_\lambda}$.

Case 1: Let $t$ be even and in $G_{T_\lambda}$. Then,
\[ D_t(G_T) = \frac{G_T}{x_t - x_{t+1}} - c \left[ \frac{2G_T}{x_t - x_{t+1}} \right] 
+ \sum_{i \text{ even, } i \neq t+1} \frac{G_T(x_t+1 - x_{i+1})}{(x_t - x_{i+1})(x_i - x_{i+1})} 
+ \sum_{i \text{ odd, } i \neq t+1} \frac{-G_T(x_t + 1 - x_i)}{(x_t - x_{i+1})(x_i - x_{i+1})} 
+ \frac{(-n + 2b - 2)G_T}{(x_t - x_{t+1})} \left[ 1 - 2c - c(n - b) - c(-n + 2b - 2) \right] \]

If we want \( D_t(G_T) = 0 \), then we need \( 1 - 2c - c(n - b) - c(-n + 2b - 2) = 0 \) and so

\[ 1 - 2c - c(n - b) - c(-n + 2b - 2) = 1 - cb = 0 \Rightarrow c = \frac{1}{b} \]

Thus, for \( t \) even, the theorem holds.

Case 2: Notice that \( D_t(-G_T) = -D_t(G_T) \). Also, if \( t \) is odd we make apply the Dunkl Operators to \(-G_T\) and we can treat \( t \) as if it were even. Thus, by Case 1 we have that the theorem holds.

Case 3: Now suppose \( t \) is not present in \( G_T \). Then:

\[ D_t(G_T) = 0 - c \left[ \sum_{\text{odd} } \frac{-G_T}{x_i - x_{i+1}} + \sum_{i \text{ even}} \frac{G_T}{x_i - x_{i+1}} \right] = 0 \]

Thus, in the third case, the value of \( c \) is irrelevant, we will always have \( D_t(G_T) = 0 \).

Since all three cases hold, we have that the theorem is true.

\[ \square \]

**Corollary 5.9.** Let \( \lambda \) be a Young diagram corresponding to \( G_T \) from theorem 5.8. If \( G_{T_{\mu}} \) is a different Garnir polynomial found from some filling of \( \lambda \) other than the one used to construct \( G_T \), then \( G_{T_{\mu}} \) is also singular.

*Proof.* From theorem 5.7 we have that the set of Garnir polynomials corresponding to \( \lambda \), \( \{G_T\} \), forms an irreducible representation of \( S_n \) and from proposition 1.5 we know that the space of singular polynomials, \( S \), forms a representation of \( S_n \).

Let \( Q = S \cap \{G_T\} \). Then \( Q \) must also be a representation of \( S_n \). Since \( \{G_T\} \) is irreducible, it follows that either \( Q = \{0\} \) or \( Q = \{G_T\} \). Since we have already shown that \( G_T \) is singular, we know \( Q \neq \{0\} \). Thus, \( Q = \{G_T\} \Rightarrow \{G_T\} \subseteq S_n \). Therefore, \( G_{T_{\mu}} \) is singular. \[ \square \]
We end our results with a conjecture closely related to the theorem just proven.

**Conjecture 5.10.** Let \( n = (b-1)q + r \) for some \( n > 0, b > 0, q > 0, r < b - 1 \). Let \( \lambda \) be the Young diagram with \( q \) rows of \( b - 1 \) boxes and one row of \( r \) boxes. Let \( T_\lambda \) be some standard Young Tableau over \( \lambda \). The Garnir polynomial associated with \( T_\lambda, G_{T_\lambda} \in \mathbb{R}[x_1, \ldots, x_n] \), is singular if and only if \( c = \frac{1}{b} \).

Also, let \( \mu \) be a Young diagram which does not conform to the above requirements and which does not correspond to the constant Garnir polynomial equal to 1. Then there does not exist any standard Young tableau, \( T_\mu \), such that \( G_{T_\mu} \) is singular.

For instance, we could have \( 18 = (6 - 1)3 + 3 \). Then we have \( b = 6, q = 3, r = 3 \), giving us the Young diagram:

```
  1 2 3 4 5 6
  7 8 9
```

Then \( G_{T_\lambda} \) is the Garnir polynomial associated to some \( T_\lambda \). The previous theorem is a special case of this conjecture and just like the previous theorem, this conjecture has an interesting relationship to division with remainder.

The following are several instances where this conjecture holds. We do not mention the Garnir polynomial corresponding to a Young diagram that is a row of \( n \) boxes, though we do list this Young diagram, since the conjecture does not comment on this Garnir polynomial and since it equals 1, so it is singular for any \( c \).

The possible Young diagrams for \( n = 4 \) are:

\[
\lambda_1 = \begin{array}{c}
\vdots \\
\end{array} \\
\lambda_2 = \begin{array}{c}
\vdots \\
\end{array} \\
\lambda_3 = \begin{array}{c}
\vdots \\
\end{array} \\
\lambda_4 = \begin{array}{c}
\vdots \\
\end{array} \\
\lambda_5 = \begin{array}{c}
\vdots \\
\end{array}
\]

Notice that \( \lambda_1, \lambda_3, \lambda_4 \) all obey the criteria set out in the conjecture, with \( b_{\lambda_1} = 2, b_{\lambda_3} = 3, b_{\lambda_4} = 4 \), while \( \lambda_2 \) does not. To determine if the conjecture holds, we use the SAGE program (see appendix).

This program first finds a Garnir polynomial corresponding to each \( \lambda_i \) and then applies the Dunkl operators to it to determine for what \( c \), if any, it is singular. Note that the Garnir polynomial found for each \( \lambda_i \) is the one found from the SYT where, if a box contains the filling \( k \), then the box immediately above it in the same column has the filling \( k + 1 \). Using the same reasoning as in corollary 5.9, if this particular Garnir polynomial is singular, then all other Garnir polynomials corresponding to \( \lambda_i \) are also singular and, similarly, if this particular Garnir polynomial is not singular for any \( c \), then all other Garnir polynomials corresponding to \( \lambda_i \) are also not singular for any \( c \).
Since the Garnir polynomial corresponding to $\lambda_1$ is the Vandermonde determinant over $n = 4$, we already know from theorem 3.5 that it is singular if and only if $c = \frac{1}{3}$ as desired.

Using the program we find:

$$D_1(G_{T_{\lambda_3}}) = -(3c - 1)(x_3 - x_4)$$
$$D_2(G_{T_{\lambda_3}}) = -(3c - 1)(x_3 - x_4)$$
$$D_3(G_{T_{\lambda_3}}) = -(3c - 1)(x_1 - x_2)$$
$$D_4(G_{T_{\lambda_3}}) = (3c - 1)(x_1 - x_2)$$

Thus, $G_{T_{\lambda_3}}$ is singular if and only if $c = \frac{1}{3}$ as desired.

And:

$$D_1(G_{T_{\lambda_4}}) = -4c + 1$$
$$D_2(G_{T_{\lambda_4}}) = 4c - 1$$
$$D_3(G_{T_{\lambda_4}}) = 0$$
$$D_4(G_{T_{\lambda_4}}) = 0$$

So $G_{T_{\lambda_4}}$ is singular if and only if $c = \frac{1}{4}$ as desired.

Finally:

$$D_1(G_{T_{\lambda_5}}) = -(5cx_1 - 3cx_2 - 3cx_3 + cx_4 - 2x_1 + x_2 + x_3)(x_2 - x_3)$$
$$D_2(G_{T_{\lambda_5}}) = -(3cx_1 - 5cx_2 + 3cx_3 - cx_4 - x_1 + 2x_2 - x_3)(x_1 - x_3)$$
$$D_3(G_{T_{\lambda_5}}) = (3cx_1 + 3cx_2 - 5cx_3 - cx_4 - x_1 - x_2 + 2x_3)(x_1 - x_2)$$
$$D_4(G_{T_{\lambda_5}}) = 0$$

Therefore, $G_{T_{\lambda_5}}$ is not singular for any $c$ and any other Garnir polynomial corresponding to $\lambda_2$ is not singular for any $c$.

For the remaining two examples, we omit the SAGE program calculations.

The possible Young diagrams for $n = 5$ are:

\[
\begin{align*}
\lambda_1 & = \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \\
\lambda_2 & = \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \\
\lambda_3 & = \begin{array}{c}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \\
\lambda_4 & = \begin{array}{c}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \\
\lambda_5 & = \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \\
\lambda_6 & = \begin{array}{c}
& & & \\
& & & \\
& & & \\
\end{array} \\
\lambda_7 & = \begin{array}{c}
& & & \\
& & & \\
\end{array} \\
\end{align*}
\]
We see that the $\lambda_i$, for $1 \leq i \leq 4$, adhere to the requirements in the conjecture while $\lambda_6$ and $\lambda_7$ do not. Furthermore, after using the SAGE program (see appendix) we see that those which do adhere to the conjecture’s requirements have singular Garnir polynomials for $c = \frac{1}{6}$ while those that do not have no Garnir polynomials that are singular for any $c$.

The possible Young diagrams for $n = 6$ are:

\[
\begin{align*}
\lambda_1 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_2 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_3 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_4 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_5 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_6 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_7 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_8 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_9 &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_{10} &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\lambda_{11} &= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{align*}
\]

The $\lambda_i$, for $1 \leq i \leq 5$, meet the requirements set out in the conjecture while $\lambda_i$, for $7 \leq i \leq 11$, do not. Furthermore, after applying the SAGE code (see Appendix), we see that the Garnir polynomials associated with the first 6 Young diagrams are singular for $c = \frac{1}{6}$ while the Garnir polynomials associated to the last 4 Young diagrams are never singular for any $c$.

References


6 Appendix

The following are two programs, written using SAGE, which we used for our calculations. The first program takes an arbitrary polynomial in \( n \) variables and of degree \( k \) and first applied the Dunkl Operators to it and then creates a matrix from which we constructed the bases for the space of singular polynomials for the instances found in section 4. The second program calculated Garnir polynomials for various SYT and then applied the Dunkl Operators to them.

**Program 1**

```python
#DUNKL CALCULATIONS SECTION

#creates a polynomial ring in n variables over a rational coefficient polynomial ring in m variables (the m variables ring will act as coefficients for variables of the n variable ring)
m = 28
A = PolynomialRing(QQ, m, 'a')
a = A.gens()
n = 7
R = PolynomialRing(A, n, 'x')
x = R.gens()

#determines the constant c and the function f to be acted on by the Dunkl Operators

C = 2/7

#creates a list of values found after each D_i acts on f

D_found = []
for i in range(n):
    rem_ind = list(range(n))
    rem_ind.remove(i)
    sD_i = 0
    for j in rem_ind:
        sD_i = sD_i + (f-f.subs({x[i]:x[j], x[j]:x[i]}))/(x[i]-x[j])
    D_i = diff(f,x[i]) - C*sD_i
    D_found.append(D_i)
```

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# displays the found values for D_i(f)
# for i in range(n):
#   D_found[i]

print('_____________________________________________________________________

# MATRIX CREATION AND ROW REDUCTION SECTION
# Creates Good variables and assigns it to True if any D_i is 0
Good = False
for i in range(n):
    if D_found[i] == 0:
        Good = True

# Matrix is only found if all D_i are not equal to 0. The program fails otherwise.
if Good == False:
    # Creates T, a list of lists where the sublists are the
    # collection of monomials in each D_i
    T = []
    for i in range(n):
        T.append(D_found[i].numerator().monomials())

    # Creates V, a list containing the coefficients of each monomial in each D_i
    V = []
    for i in range(n):
        for j in range(len(T[i])):
            V.append(D_found[i].numerator().monomial_coefficient(T[i][j]))

    # Creates W, a list of lists where each sublist is the
    # collection of coefficients for the
    # coefficient sum of each monomial in each D_i
    W = []
    for i in range(len(V)):
        U = []
        for j in range(m):
            U.append(V[i].monomial_coefficient(a[j]))
        W.append(U)

    # Creates a matrix Q which has, as its rows, the coefficients of the coefficient sum
    # of each monomial in each D_i
    G1 = matrix(W[0])
    M = G1.transpose()
    for i in range(1, len(W)):
        G2 = vector(W[i])
        M = M.augment(G2)

    Q = M.transpose()
    QF = Q.echelon_form()
rowcount = QF.nrows()
colcount = QF.ncols()
for i in range(rowcount):
    AllZero = True
    for j in range(colcount):
        if QF.row(i)[j] != 0:
            AllZero = False
    if AllZero == False:
        QF.row(i)

Program 2

n = 10
R = PolynomialRing(QQ, n, 'x')
x = R.gens()
#c = var('c')

# partition

lam = [2,2,2]

# Garnir polynomial for lambda
f = 1
l=len(lam)
tmp = 0
for k in range(l):
    w=prod([prod([x[i]-x[j] for j in range(i+1,tmp+lam[k])])
            for i in range(tmp,tmp+lam[k])])
    f=f*w
    tmp=tmp+lam[k]

#Dunkl operators

f = (x[0]-x[1])*(x[2]-x[3])*(x[4]-x[5])*(x[6]-x[7])*(x[8]-x[9])
c = 1/6
factor(f)
D_found = []
for i in range(n):
    rem_ind = list(range(n))
    rem_ind.remove(i)
    sD_i = 0
    for j in rem_ind:
        sD_i = sD_i + (f-f.subs({x[i]:x[j],x[j]:x[i]}))/(x[i]-x[j])
    D_i = diff(f,x[i]) - c*sD_i
    D_found.append(D_i)
for i in range(n):
    if D_found[i] != 0:
        factor(D_found[i])
    else:
        D_found[i]