Math 150c: Modern Algebra
Final Exam

1. If M is a module over a ring R, then an element $r \in R$ is said to *annihilate* M if $rm = 0$ for all $m \in M$. Consider the plane $\mathbb{R}^2$ as a module over $\mathbb{R}[x]$ with x acting by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Find a non-zero element of $\mathbb{R}[x]$ that annihilates this module.

2. If a group has 64 elements, what are the possible dimensions of its complex irreducible representations? (For example, can it have a 57-dimensional irrep?) You do not have to produce examples for the dimensions that you think are possible; just combine all of the restrictions that are available from established theorems.

3. Let G be a finite group, and let $A = \frac{1}{|G|}\sum_{g \in G} g$ be the averaging element in the group algebra $\mathbb{C}[G]$. Is this element always, sometimes, or never in the center of $\mathbb{C}[G]$?

4. For each positive integer n, the nth root $\sqrt[n]{5}$ may or may not be constructible using the field operations $(+,-,\times,\div)$, square roots, and integers. Construct $\sqrt[n]{5}$ for those n for which this is possible. For the others, explain why it is not constructible. (You can use the fact that $x^n - 5$ is an irreducible polynomial for every n.)

5. The polynomial $x^8 + x^5 + x^3 + x \in \mathbb{F}_2[x]$ vanishes identically as a function from $\mathbb{F}_2$ to $\mathbb{F}_2$. (That is, both of its values are zero.) Show that there is a larger field $\mathbb{F}_q$ with $q = 2^k$ such that the same polynomial does not vanish as a function from $\mathbb{F}_q$ to $\mathbb{F}_q$.

6. Recall that $\omega = e^{2\pi i/7}$ has minimal polynomial $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ over $\mathbb{Q}$. Using this fact, find the minimal polynomial of $\cos\frac{2\pi}{7}$.

7. Find the dimension of the field $\mathbb{Q}(\sqrt{1-\sqrt{3}})$ as a vector space over $\mathbb{Q}$.

8. Does the polynomial $x^4 + 2x^3 - x^2 - 2x + 1$ have any repeated roots? (Read it as a polynomial over $\mathbb{Q}$ with roots in $\mathbb{C}$.)