Winter 2003 Mathematics Graduate Program Preliminary Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Analysis

Problem 1. Prove or disprove: Any linear bounded operator in a complex Hilbert space can be written as a linear combination of two self-adjoint operators. (Hint: Consider first the finite-dimensional case.)

Problem 2. Consider the Hilbert space $L^2[-1, 1]$.
   (i) Find the orthogonal complement of the space of all polynomials. (Hint: Use the Stone-Weierstrass theorem.)
   (ii) Find the orthogonal complement of the space of polynomials in $x^2$.

Problem 3. Consider the space of all polynomials on $[0, 1]$ vanishing at the origin, with the sup norm. Prove that the space is not complete and find its completion.

Problem 4. Prove that $\mathbb{R}^1$ with the metrics
   (i) $\rho(x, y) = |\arctan(x) - \arctan(y)|$
   or
   (ii) $\rho(x, y) = |\exp(x) - \exp(y)|$
   is incomplete, and find the completion in each case.

Problem 5. Consider a continuous mapping of the closed unit square $[0, 1] \times [0, 1]$ into some metric space $X$. Prove that the image of the square under such a mapping is compact.

Problem 6. Prove or disprove:
   $C[0, 1]$ with the usual sup norm is a Hilbert space. (Hint: Consider two continuous functions with disjoint supports and calculate the norm of their sum.)
2. Algebra and Linear Algebra

Problem 7. Suppose that $A$ and $B$ are complementary subgroups in a group $G$, meaning that $G = AB$ and that $A$ and $B$ intersect trivially (but perhaps neither $A$ nor $B$ is normal). Show that each right coset of $A$ intersects each left coset of $B$ in exactly one element.

Problem 8. Find all automorphisms of $\mathbb{Z}[x]$, the ring of polynomials over $\mathbb{Z}$.

Problem 9. Let $R$ be a commutative ring with identity and prime characteristic $p$. Show that the map

$$\varphi : R \rightarrow R$$

$$r \mapsto r^p$$

is a homomorphism of rings (it's called the Frobenius homomorphism).

Problem 10. Find a subgroup of the unit quaternions $Q$ which is a circle. Argue a corollary: The 3-sphere is the union of disjoint circles.

Problem 11. Let $G$ be a group and let $H$ be a subgroup of $G$ with finite index $n > 1$.

a. Show that the map $G \times G/H \rightarrow G/H$ defined by $(g, aH) \mapsto gaH$ gives an action of $G$ on the space $G/H$ of left cosets of $H$ in $G$.

b. Show that if, in addition, $G$ is finite and the order of $G$ does not divide $n!$, then $G$ is not simple.

c. Can a group of order $2^2 \cdot 3 \cdot 19^2$ be simple?

Problem 12. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. Find a matrix $B$ so that $BAB^{-1}$ is diagonal.