Fall 2013: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Let $G \subset M_n(\mathbb{C})$ be a group of complex $n \times n$ matrices. Let V be the linear span of G, and let $V^\times$ be the set of invertible elements of V. Show that $V^\times$ is also a group.

Problem 2: Consider an attempt to make an $\mathbb{R}$-linear map

$$f : \mathbb{C} \otimes_\mathbb{C} \mathbb{C} \to \mathbb{C} \otimes_\mathbb{R} \mathbb{C} \qquad \text{or} \qquad \mathbb{C} \otimes_\mathbb{R} \mathbb{C} \to \mathbb{C} \otimes_\mathbb{C} \mathbb{C},$$

in either direction given by the formula

$$f(x \otimes y) = x \otimes y.$$

In which direction is this map well-defined? Is it then surjective? Is it injective?

Problem 3: The dihedral group D_4 acts as the symmetries of a square in the plane $\mathbb{R}^2$ with coordinates x and y. Suppose that the corners of this square are at $(\pm 1, \pm 1)$. Then D_4 acts linearly, and it therefore has an induced action on the vector space V_n of homogeneous polynomials in x and y of degree n. Find the character of V_n viewed as a representation of D_4. (Note: the character will depend on n.)

Problem 4: Let G be a group with an odd number of elements that has a normal subgroup N with 17 elements. Show that N lies in the center of G.

Problem 5: Is it possible to have a field extension $F \subseteq K$ with $[K : F] = 2$, where both fields F and K are isomorphic to the field $\mathbb{Q}(x)$?

Problem 6: Compute $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.