Winter 2007: MA Algebra Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let R be a commutative ring with identity, and let I be an ideal of R. Under what conditions on I is R/I a field? An integral domain? A commutative ring with identity?

Problem 2. Let V be a vector space, and let A and B be a pair of commuting operators on V. Show that if W is an invariant subspace for A, then so are the spaces BW and $B^{-1}W := \{ v \in V \ : \ Bv \in W \}$.

Problem 3. Suppose the group G has character table

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & \zeta_5^3 + \zeta_5^2 + 1 & \zeta_5^4 + \zeta_5 + 1 \\ 3 & -1 & 0 & \zeta_5^4 + \zeta_5 + 1 & \zeta_5^3 + \zeta_5^2 + 1 \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0, \end{array}$$

where ζ_5 is a primitive 5-th root of unity (so $\zeta_5^4 + \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 = 0$).

(a) Prove that G is a simple group of order 60, and determine the sizes of its conjugacy classes.

(b) How does the tensor product of the two 3-dimensional irreps decompose into irreducibles?

Problem 4. Suppose that the group G is generated by elements x and y that satisfy $x^5y^3 = x^8y^5 = 1$. Is G the trivial group?

Problem 5. Let R be a principal ideal domain and $I \subset R$ an ideal. Prove that every ideal in the quotient ring R/I is a principal ideal. Show that R/I is not necessarily a principal ideal domain.

Problem 6.

(a) Give an example of a 4×4 complex matrix having only one eigenvalue, equal to 3, with the space of eigenvectors having dimension 2.

(b) Let us consider the set K of all matrices obeying the conditions of (a). The group $GL_4(\mathbb{C})$ acts on K by means of the transformations $\phi_A(X) = AXA^{-1}$. How many orbits does this action have?

Winter 2007: MA Analysis Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0,1]) \to C([0,1])$ be a bounded linear operator. Suppose that $\|S(p)\| \le 2$ for all polynomials p. Show that S is the zero operator.

Problem 2. For $p \ge 1$, let $l^p(\mathbb{N})$ be the set of sequences (x_n) such that

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

(a) Show that if $1 \le p < q < \infty$ then $l^p(\mathbb{N}) \subseteq l^q(\mathbb{N})$.

(b) Show that if $1 \le p < q < \infty$ then $l^p(\mathbb{N}) \ne l^q(\mathbb{N})$.

Problem 3. Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y);$$

in particular, both limits exist. Does it follow that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exists?

Problem 4. Let X be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < Cd(x,y)$ for all $x \ne y$. The function f is said to be a *weak contraction* if $d(f(x), f(y)) < d(x,y)$ for all $x \ne y$, without the constant C. The contraction mapping theorem says that if f is a contraction, then it has a fixed point. Show that the theorem also holds when f is a weak contraction and X is compact.

Problem 5. Construct the Green's function for the Dirichlet boundary-value problem

$$-u'' + 4u = f, \quad u(0) = u(2) = 0.$$

Problem 6. Let U be a unitary operator on a Hilbert space. Prove that the spectrum of U lies on the unit circle.