Winter 2007: MA Algebra Preliminary Exam

Instructions:

1. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1. Let \( R \) be a commutative ring with identity, and let \( I \) be an ideal of \( R \). Under what conditions on \( I \) is \( R/I \) a field? An integral domain? A commutative ring with identity?

Problem 2. Let \( V \) be a vector space, and let \( A \) and \( B \) be a pair of commuting operators on \( V \). Show that if \( W \) is an invariant subspace for \( A \), then so are the spaces \( BW \) and \( B^{-1}W := \{ v \in V : Bv \in W \} \).

Problem 3. Suppose the group \( G \) has character table

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
3 & -1 & 0 & \zeta_5^3 + \zeta_5^2 + 1 & \zeta_5^4 + \zeta_5 + 1 \\
3 & -1 & 0 & \zeta_5^4 + \zeta_5 + 1 & \zeta_5^3 + \zeta_5^2 + 1 \\
4 & 0 & 1 & -1 & -1 \\
5 & 1 & -1 & 0 & 0,
\end{array}
\]

where \( \zeta_5 \) is a primitive 5-th root of unity (so \( \zeta_5^4 + \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 = 0 \)).

(a) Prove that \( G \) is a simple group of order 60, and determine the sizes of its conjugacy classes.

(b) How does the tensor product of the two 3-dimensional irreps decompose into irreducibles?

Problem 4. Suppose that the group \( G \) is generated by elements \( x \) and \( y \) that satisfy \( x^5y^3 = x^5y^5 = 1 \). Is \( G \) the trivial group?

Problem 5. Let \( R \) be a principal ideal domain and \( I \subset R \) an ideal. Prove that every ideal in the quotient ring \( R/I \) is a principal ideal. Show that \( R/I \) is not necessarily a principal ideal domain.

Problem 6.

(a) Give an example of a 4 \( \times \) 4 complex matrix having only one eigenvalue, equal to 3, with the space of eigenvectors having dimension 2.

(b) Let us consider the set \( K \) of all matrices obeying the conditions of (a). The group \( GL_4(\mathbb{C}) \) acts on \( K \) by means of the transformations \( \phi_A(X) = AXA^{-1} \). How many orbits does this action have?
Winter 2007: MA Analysis Preliminary Exam

Instructions:

(1) Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

(2) Use separate sheets for the solution of each problem.

Problem 1. Let $C([0, 1])$ be the Banach space of continuous real-valued functions on $[0, 1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0, 1]) \to C([0, 1])$ be a bounded linear operator. Suppose that $\|S(p)\| \leq 2$ for all polynomials $p$. Show that $S$ is the zero operator.

Problem 2. For $p \geq 1$, let $l^p(N)$ be the set of sequences $(x_n)$ such that

$$
\| (x_n) \|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.
$$

(a) Show that if $1 \leq p < q < \infty$ then $l^p(N) \subseteq l^q(N)$.

(b) Show that if $1 \leq p < q < \infty$ then $l^p(N) \neq l^q(N)$.

Problem 3. Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$
\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} f(x, y);
$$
in particular, both limits exist. Does it follow that

$$
\lim_{(x, y) \to (0, 0)} f(x, y)
$$
exists?

Problem 4. Let $X$ be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < Cd(x, y)$ for all $x \neq y$. The function $f$ is said to be a weak contraction if $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, without the constant $C$. The contraction mapping theorem says that if $f$ is a contraction, then it has a fixed point. Show that the theorem also holds when $f$ is a weak contraction and $X$ is compact.

Problem 5. Construct the Green's function for the Dirichlet boundary-value problem

$$
-u'' + 4u = f, \quad u(0) = u(2) = 0.
$$

Problem 6. Let $U$ be a unitary operator on a Hilbert space. Prove that the spectrum of $U$ lies on the unit circle.