Fall 2009: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. For $\epsilon > 0$, let η_ϵ denote the family of *standard* mollifiers on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the functions

$$u_\epsilon = \eta_\epsilon * u \text{ in } \mathbb{R}^2 .$$

Prove that

$$\epsilon \|Du_\epsilon\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)} ,$$

where the constant C depends on the mollifying function, but not on u.

Problem 2.
Let $B(0,1) \subset \mathbb{R}^3$ denote the unit ball $\{|x| < 1\}$. Prove that $\log|x| \in H^1(B(0,1))$.

Problem 3. Prove that the continuous functions of compact support are a dense subspace of $L^2(\mathbb{R}^d)$.

Problem 4. There are several senses in which a sequence of bounded operators $\{T_n\}$ can converge to a bounded operator T (in a Hilbert space $\mathcal{H}$). First, there is convergence in the norm, that is, $\|T_n - T\| \to 0$, as $n \to \infty$. Next, there is a weaker convergence, which happens to be called strong convergence, that requires that $T_n f \to Tf$, as $n \to \infty$, for every vector $f \in \mathcal{H}$. Finally, there is weak convergence that requires $(T_n f, g) \to (Tf, g)$ for every pair of vectors $f, g \in \mathcal{H}$.

(a) Show by examples that weak convergence does not imply strong convergence, nor does strong convergence imply convergence in norm.

(b) Show that for any bounded operator T there is a sequence $\{T_n\}$ of bounded operators of finite rank so that $T_n \to T$ strongly as $n \to \infty$.

Problem 5. Let $\mathcal{H}$ be a Hilbert space. Prove the following variants of the spectral theorem.

(a) If T_1 and T_2 are two linear symmetric and compact operators on $\mathcal{H}$ that commute (that is, $T_1T_2 = T_2T_1$), show that they can be diagonalized sim ultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.

(b) A linear operator on $\mathcal{H}$ is *normal* if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized.

(c) If U is unitary, and $U = \lambda I - T$, where T is compact, then U can be diagonalized.

Problem 6. Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.