Graduate Group in Applied Mathematics  
University of California, Davis  
**Preliminary Exam**  
September 21, 2010

**Instructions:**

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

**Problem 1:** (10 points)  
Let \( f(x, y) \) denote a \( C^1 \) function on \( \mathbb{R}^2 \). Suppose that  
\[ f(0, 0) = 0. \]

Prove that there exist two functions, \( A(x, y) \) and \( B(x, y) \), both continuous on \( \mathbb{R}^2 \) such that  
\[ f(x, y) = xA(x, y) + yB(x, y) \quad \forall (x, y) \in \mathbb{R}^2 \]

(Hint: Consider the function \( g(t) = f(tx, ty) \) and express \( f(x, y) \) in terms of \( g \) via the fundamental theorem of calculus.)

**Problem 2:** (10 points)  
The Fourier transform \( \mathcal{F} \) of a distribution is defined via the duality relation  
\[ \langle \mathcal{F} f, \phi \rangle = \langle f, \mathcal{F}^* \phi \rangle \]

for all \( \phi \in C_0^\infty(\mathbb{R}) \), the smooth compactly-supported test functions on \( \mathbb{R} \), where  
\[ \mathcal{F}^* \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \phi(\xi) \, d\xi. \]

Explicitly compute \( \mathcal{F} f \) for the function  
\[ f(x) = \begin{cases} 
  x, & x > 0 \\
  0, & x \leq 0
\end{cases}. \]
**Problem 3:** (10 points)

Let \( \{ P_n(x) \}_{n=1}^{\infty} \) denote a sequence of polynomials on \( \mathbb{R} \) such that

\[
P_n \to 0 \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty.
\]

Prove that, for \( n \) sufficiently large, all \( P_n \) are constant polynomials.

**Problem 4:** (10 points)

For \( g \in L^1(\mathbb{R}^3) \), the convolution operator \( G \) is defined on \( L^2(\mathbb{R}^3) \) by

\[
Gf(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} g(x - y)f(y) \, dy, \quad f \in L^2(\mathbb{R}^3).
\]

Prove that the operator \( G \) with

\[
g(x) = \frac{1}{\frac{4\pi}{|x|}} \quad x \in \mathbb{R}^3,
\]

is a bounded operator on \( L^2(\mathbb{R}^3) \), and the operator norm \( \|G\|_{op} \leq 1 \).
Problem 5: (10 points)
Consider the map which associates to each sequence \( \{x_n : n \in \mathbb{N}, x_n \in \mathbb{R}\} \) the sequence, \( \{(F(\{x_n\}))_m ; m \in \mathbb{N}, (F(\{x_n\}))_m \in \mathbb{R}\} \), defined as follows:

\[
\left\{ F(\{x_n\}) \right\}_m \coloneqq \frac{x_m}{m} \quad \text{for} \quad m = 1, 2, \ldots
\]

1. Determine (with proof) the values of \( p \in [1, \infty] \) for which the map \( F : l^p \to l^1 \) is well-defined and continuous.

2. Next, determine the values of \( q \in [1, \infty] \) for which the map \( F : l^q \to l^2 \) is well-defined and continuous.

Note for \( 1 \leq p < \infty, l^p \) denotes the space of sequences \( \{x_n\}_{n=1}^{\infty} \) such that \( \sum_{n=1}^{\infty} |x_n|^p < \infty \), while \( l^\infty \) denotes the space of sequences \( \{x_n\}_{n=1}^{\infty} \) such that \( \sup_{n \in \mathbb{N}} |x_n| < \infty \).

Problem 6: (10 points)
For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof, e.g., by citing a relevant theorem; if false, give a counterexample.

Let \( H \) denote a separable Hilbert space and \( (x_n) \) a sequence of \( H \).

(a) If \( (x_n) \) is weakly convergent then it is strongly convergent.

(b) If \( (x_n) \) is strongly convergent then it is bounded.

(c) If \( (x_n) \) is weakly convergent then it is bounded.

(d) If \( (x_n) \) is bounded, there exists a strongly convergent subsequence of \( (x_n) \).

(e) If \( (x_n) \) is bounded, there exists a weakly convergent subsequence of \( (x_n) \).

(f) If \( (x_n) \) is weakly convergent and \( T \) is a bounded linear operator from \( H \) to \( \mathbb{R}^d \), for some \( d \), then \( T(x_n) \) converges in \( \mathbb{R}^d \).
Problem 7: (10 points)
Consider the first order ordinary differential equation
\[ \frac{dx}{dt} = \beta + \alpha x - x^3 = f(x), \]
with \( \alpha, \beta \in \mathbb{R}. \)

(a) What conditions on \( f(x) \) and \( f'(x) \) must be satisfied simultaneously at bifurcation points? \textit{Briefly} explain your answers.

(b) Use these conditions to find the curves of bifurcation points in \( \alpha \) vs. \( \beta \) parameter space? Sketch the corresponding curves in the \( \alpha \) vs. \( \beta \) plane.

(c) Sketch the following bifurcation diagrams. Indicate the stability of the fixed points on the diagrams and classify the bifurcations that occurs. A detailed analytical treatment of the system is not required, but some justification (e.g., graphical arguments) of your answers is required.
(i) Use \( \alpha \) as the bifurcation parameter and hold \( \beta \) constant at \( \beta = 0. \)
(ii) Use \( \alpha \) as the bifurcation parameter and hold \( \beta \) constant with \( \beta > 0. \)
(iii) Use \( \beta \) as the bifurcation parameter and hold \( \alpha \) constant with \( \alpha > 0. \)

Problem 8: (10 points)
Consider the system of ordinary differential equations in polar coordinates
\[ \frac{dr}{dt} = r(1-r^2)(4-r^2), \quad r \geq 0 \]
\[ \frac{d\theta}{dt} = 2 - r^2. \]

Sketch the phase-portrait. Label all fixed points and limit cycles, and indicate their stability.