Spring 2013: PhD Analysis Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

**Problem 1:** Consider the Hilbert space $H = L^2([0,1])$ with inner product $\langle f, h \rangle = \int_0^1 f(x)h(x) \, dx$. Let

$$V = \{ f \in L^2([0,1]) : \int_0^1 xf(x) \, dx = 0 \} \subset H$$

and $g(x) \equiv 1$. Find the closest element to $g$ in $V$. Justify your answer.

**Problem 2:** Let $(B, \| \|)$ be a Banach space. Recall that the spectrum of a bounded linear operator $A \in \mathcal{L}(B)$ is defined as

$$\sigma(A) = \{ \lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible} \}.$$

Consider a sequence of bounded linear operators $A_n \in \mathcal{L}(B)$ which converges in norm to a bounded linear operator $A \in \mathcal{L}(B)$. Assume that all spectra are the same, i.e. $\sigma_0 := \sigma(A_1) = \sigma(A_2) = \ldots$. Show that $\sigma_0 \subset \sigma(A)$.

**Problem 3:** Consider the function

$$f(x) = \begin{cases} 2\sin(x) + 3, & x > 0 \\ -2\sin(x) + c, & x \leq 0 \end{cases}.$$

Find its distributional derivative $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$. For which values of $c$ will $f \in W^{1,p}_{\text{loc}}(\mathbb{R})$? Justify your answer.

**Problem 4:** Prove that

$$\lim_{\epsilon \to 0^+} \int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin(1/x) \, dx = 0.$$
**Problem 5:** Prove that the image of the space $C^k(T)$ of $k$ times continuously differentiable functions on the unit circle under the Fourier transform is contained in the set of sequences satisfying $|c_n| = o(|n|^{-k})$ and contains the set of sequences satisfying $|c_n| = o(|n|^{-k-1-\epsilon})$, $\epsilon > 0$.

(Recall: $f(n) = o(h(n))$ as $n \to \infty$ means that for every $\delta > 0$ there exists an $N$ such that $|f(n)| \leq \delta |g(n)|$ for all $n > N$).

**Problem 6:** Let $I$ be the interval $(0,1)$ and $q \geq p \geq 1$. Show that there exists a constant $C = C(p,q,I)$ such that

$$\|u\|_{L^q(I)} \leq C \|u\|_{W^{1,p}(I)}$$

for all $u \in W_0^{1,p}(I)$. (Hint: First show that $\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}$ for all $u \in W_0^{1,p}(I)$.)