Instructions:

• This exam has 4 pages (8 problems) and is closed book.

• The first 6 problems cover Analysis and the last 2 problems cover ODEs.

• All problems are worth 10 points.

• Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

• Use separate sheets for the solution of each problem.

Problem 1: (10 points)
Let $\Omega = (0, 1)$, the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = ne^{-nx}$. Prove that $f_n \not\rightharpoonup f$ weakly in $L^1(\Omega)$, i.e., the sequence $f_n$ does not converge in the weak topology of $L^1(\Omega)$.
(Hint: Prove by contradiction.)

Problem 2: (10 points)
Let $\Omega = (0, 1)$, and consider the linear operator $A = -\frac{d^2}{dx^2}$ acting on the Sobolev space of functions $X$ where

$$X = \left\{ u \in H^2(\Omega) \mid u(0) = 0, u(1) = 0 \right\},$$

and where

$$H^2(\Omega) = \left\{ u \in L^2(\Omega) \mid \frac{du}{dx} \in L^2(\Omega), \frac{d^2u}{dx^2} \in L^2(\Omega) \right\}.$$

Find all of the eigenfunctions of $A$ belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) \mid \alpha \in \mathbb{R}\},$$

as well as their corresponding eigenvalues.
Problem 3: (10 points)
Let $\Omega = (0, 1)$, the open unit interval in $\mathbb{R}$, and set

$$v(x) = (1 + |\log x|)^{-1}.$$ 

Show that $v \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $\frac{v}{x} \not\in L^1(\Omega)$. (This shows the failure of Hardy’s inequality in $L^1$.)

Note that $W^{1,1}(\Omega) = \{u \in L^1(\Omega) \mid \frac{du}{dx} \in L^1(\Omega)\}$, where $\frac{du}{dx}$ denotes the weak derivative.

Problem 4: (10 points)
Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period $2\pi$. Show that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n \tau_n \delta \text{ in } \mathcal{D}'$$ \hspace{1cm} (1)

that is, that equality in equation (1) holds in the sense of distributions, and relate $b_n$ to the coefficients of the Fourier series. Note that $\delta$ denotes the Dirac distribution and $\tau_y$ is the translation operator, given by $\tau_y f(x) = f(x + y)$.

(Hint: Write $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with convergence in $L^2(0, 2\pi)$ and where the coefficients $c_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-inx} f(x) \, dx$.)
Problem 5: (10 points)
Let \( f(x) \) be a periodic continuous function on \( \mathbb{R} \) with period \( 2\pi \). Given \( \epsilon > 0 \), prove that for \( N < \infty \) there is a finite Fourier series
\[
\phi(x) = a_0 + \sum_{n=1}^{N} \left[ a_n \cos(nx) + b_n \sin(nx) \right]
\]
such that
\[
|\phi(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R}.
\]
This shows that the space of real-valued trigonometric polynomials on \( \mathbb{R} \) (functions which can be expressed as in (2)) are \textit{uniformly} dense in the space of periodic continuous function on \( \mathbb{R} \) with period \( 2\pi \).

\textbf{(Hint:} The Stone-Weierstrass theorem states that if \( X \) is compact in \( \mathbb{R}^d \), \( d \in \mathbb{N} \), then the algebra of all real-valued polynomials on \( X \) (with coordinates \( (x_1, x_2, ..., x_d) \)) is dense in \( C(X) \). )

Problem 6: (10 points)
For \( \alpha \in (0, 1] \), the space of Hölder continuous functions on the interval \([0, 1]\) is defined as
\[
C^{0, \alpha}(\mathbb{R}):= \{ u \in C([0,1]) : |u(x) - u(y)| \leq C|x - y|^{\alpha}, x, y \in [0, 1]\},
\]
and is a Banach space when endowed with the norm
\[
\|u\|_{C^{0, \alpha}(\mathbb{R})} = \sup_{x \in [0,1]} |u(x)| + \sup_{x, y \in [0,1]} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.
\]
Prove that the closed unit ball \( \{ u \in C^{0, \alpha}([0, 1]) : \|u\|_{C^{0, \alpha}([0, 1])} \leq 1 \} \) is a compact set in \( C([0,1]) \).

\textbf{(Hint:} The Arzela-Ascoli theorem states that if a family of continuous functions \( U \) is equicontinuous and uniformly bounded on \([0, 1]\), then each sequence \( u_n \) in \( U \) has a uniformly convergent subsequence. Recall that \( U \) is uniformly bounded on \([0, 1]\) if there exists \( M > 0 \) such that \( |u(x)| < M \) for all \( x \in [0, 1] \) and all \( u \in U \). Further, recall that \( U \) is equicontinuous at \( x \in [0, 1] \) if given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |u(x) - u(y)| < \epsilon \) for all \( |x - y| < \delta \) and every \( u \in U \).)
Problem 7: (10 points)
Consider the system of ordinary differential equations
\[
\frac{dx}{dt} = -x(y + 1)
\]
\[
\frac{dy}{dt} = 1 - x^2 - y^2
\]
(a) Show that \((x, y) = (0, 1)\) and \((0, -1)\) are fixed points of the system. Linearize the system about the fixed points \((0, 1)\) and \((0, -1)\) and use linearized system to classify the fixed points.
(b) Sketch the phase portrait of the full system and re-classify the fixed points.

Problem 8: (10 points)
Consider the system describing a particle mass moving in a double-well potential \(V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4\), i.e.,
\[
\ddot{x} = -\frac{dV}{dx} = x - x^3.
\]
(a) Show that the energy \(E(x, \dot{x}) = \frac{\dot{x}^2}{2} + V(x)\) is a conserved quantity for this system, i.e. \(E(x, \dot{x})\) is constant along trajectories.
(b) Sketch the \(x, \dot{x}\)-phase portrait. Classify the fixed points of the system \((0, 0)\) and \((\pm 1, 0)\).