Fall 2011: MA Algebra Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1:
Show that there is no commutative ring with the identity whose additive group is isomorphic to \( \mathbb{Q}/\mathbb{Z} \).

Problem 2:
Let \( p \neq 2 \) be prime and \( F_p \) be the field of \( p \) elements.
(a) How many elements of \( F_p \) have square roots in \( F_p \)?
(b) How many have cube roots in \( F_p \)?

Problem 3:
Prove that every finite group is isomorphic to a certain group of permutations (a subgroup of \( S_n \) for some \( n \)).

Problem 4:
Let \( G \) be the subgroup of \( S_{12} \) generated by \( a = (1 \ 2 \ 3 \ 4 \ 5 \ 6)(7 \ 8 \ 9 \ 10 \ 11 \ 12) \) and \( b = (1 \ 7 \ 4 \ 10)(2 \ 12 \ 5 \ 9)(3 \ 11 \ 6 \ 8) \). Find the order of \( G \), the number of conjugacy classes of \( G \), and the character table of \( G \).

Problem 5:
Prove or disprove: If the group \( G \) of order 55 acts on a set \( X \) of 39 elements then there is a fixed point.

Problem 6:
Prove or disprove: \( (\mathbb{Z}/35\mathbb{Z})^* \cong (\mathbb{Z}/39\mathbb{Z})^* \cong (\mathbb{Z}/45\mathbb{Z})^* \cong (\mathbb{Z}/70\mathbb{Z})^* \cong (\mathbb{Z}/78\mathbb{Z})^* \cong (\mathbb{Z}/90\mathbb{Z})^* \). Here \( (\mathbb{Z}/n\mathbb{Z})^* \) is the group of units in \( \mathbb{Z}/n\mathbb{Z} \).