MAT 22A : Linear algebra
December 13, 2013

1. (a) Compute the projection of the vector \[
\begin{pmatrix}
1 \\
2 \\
3 \\
0
\end{pmatrix}
\]
to the linear subspace \(X = \{x \in \mathbb{R}^4 \mid x_1 = x_2 \text{ and } x_3 + x_4 = 0\}\).

(b) Over a flat and long drive, a driver manages to link his car’s gas mileage to his average speed. In particular he collects 4 data given as follows

<table>
<thead>
<tr>
<th>Trip Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average speed (mph)</td>
<td>60</td>
<td>64</td>
<td>62</td>
<td>60</td>
</tr>
<tr>
<td>Gas mileage (mpg)</td>
<td>50</td>
<td>47</td>
<td>49</td>
<td>51</td>
</tr>
</tbody>
</table>

Find the best linear relation that links the gas mileage to the average speed.

**Hint**: To simplify the computation, try to fit the relation \((\text{mpg} - 50) = C + D(\text{mph} - 60)\).
2. Consider the matrix

\[
A = \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
2 & 2 & 0
\end{pmatrix}
\]

(a) Prove that the columns of \( A \) are linearly independent.

(b) Compute an orthogonal basis of \( \text{C}(A) \), the column space of \( A \).

(c) Compute a vector that is in the orthogonal complement of \( \text{C}(A) \).
3. Let \( A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 1 & -\frac{1}{4} & 1 \\ \frac{1}{2} & -\frac{3}{4} & \frac{3}{2} \end{pmatrix} \). 

(a) Prove that the rank of \( A \) is 2.

(b) Prove that \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) is an eigenvector of \( A \).

(c) Compute all eigenvalues and eigenvectors of \( A \).

(d) Compute \( A^{200} \) with 5 significant digits.
4. Give a short answer and a justification to the following questions. The justification is more important than a correct answer.

(a) What is the determinant of
\[
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & 0 \\
1 & 0 & 2 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix}
\]?

(b) True or false? A matrix is diagonalizable if and only if, for every eigenvalue, the geometric multiplicity is less than or equal to the algebraic multiplicity.

(c) Given \( n \) pairs \((a_i, b_i), i = 1, \ldots, n\), explain how to fit a relation of the type \( b \approx Ca^D \) using linear least-squares.

(d) True or false? If \( A \) has eigenvalues 1, 2, 3 then \( A^{-1} \) has eigenvalues \( \frac{1}{3}, \frac{1}{2}, \frac{1}{3} \).

(e) True or false? \{\((1, 0, 0), (0, 1, 0), (0, 0, 1)\}\} is an orthonormal basis for the row space of
\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & 3 \\
0 & -1 & 1
\end{pmatrix}
\].

(f) True or false? If \( Q \) is an orthogonal matrix, then \( Q^T \) is an orthogonal matrix too.