Q1______ Last Initial ________________________________
Q2______ Section _________________________________
Q3______ Student ID _____________________________
Q4______ FULL Name _____________________________
Q5______
Q6______
Σ ______

FINAL EXAMINATION III

21D §C01-06, 6:00-8:00 pm
Wednesday December 9, 2015

Declaration of honesty: I, the undersigned, do hereby swear to uphold the very highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, peeking at the person next to me whose answer is probably wrong anyway, books, calculators, mobile phones, blackberries, blueberries, boysenberries, raspberries, artificial intelligence or any other electronic device. Volcanic-emotional-support-pet rocks without tattoos permitted.

Signature ___________________ Date ______________
Q1 scratch/extra space (do not erase your scratch computations, they might earn partial credit):
Question 1

The electric field of a particle at the origin with charge $q$ is given by

$$
\frac{q}{4\pi |\vec{r}|^3}.
$$

The electric field $\vec{E}$ of two charged particles at two different positions is obtained by adding the electric field of the first particle to the electric field of the second particle. Write down the electric field of a pair of particles where the first particle has charge $q$ and is at the origin and the second particle has charge $Q$ and is at position $\vec{a}$. Compute the divergence of this electric field. Draw a picture showing the two particles and four closed surfaces $S_q$, $S_Q$, $S_{Qq}$ and $S$, where $S_q$ contains the charge $q$, $S_Q$ contains the charge $Q$, $S_{Qq}$ contains both charges and $S$ contains neither charge. Calculate the flux integral of $\vec{E}$ for each of these four surfaces. Well-organized and explained responses will receive more credit.
Q2 scratch/extra space (do not erase your scratch computations, they might earn partial credit):
Let $D \subset \mathbb{R}^3$ be a region in space. A function $f : D \to \mathbb{R}$ is said to be *harmonic* if it obeys
\[
\Delta f = 0,
\]
where $\Delta f = \nabla \cdot \nabla f$.

(i) Suppose $f$ is harmonic throughout $D$ and the boundary of $D$ is given by a closed surface $S = \partial D$. **Show** that
\[
\oint_S dA \hat{N} \cdot \nabla f = 0.
\]

(ii) **Show** that if $f$ is harmonic throughout $D$, then
\[
\oint_S dA \hat{N} \cdot (f \nabla f) = \int_D dV |\nabla f|^2.
\]
Q3 scratch/extra space (do not erase your scratch computations, they might earn partial credit):
Question 3
Suppose you are given a vector that depends smoothly on two parameters, i.e. \( \vec{r} = \vec{r}(\alpha, \beta) \). Moreover you know that \( \vec{r}(0, 0) = j + 2k \). Use calculus to write down an approximation for \( \vec{r}(0.01, 0.003) \). Find approximately the area of the parallelogram with corners labeled by the vectors \( \vec{r}(0, 0), \vec{r}(0.003), \vec{r}(0.01, 0) \) and \( \vec{r}(0.01, 0.003) \). Apply both your formulæ to the case

\[
\vec{r} = \alpha^2 i + (1 + \alpha + \beta) j + (\alpha \beta + \beta + 2) k.
\]
Q4 scratch/extra space (do not erase your scratch computations, they might earn partial credit):
Question 4
You probably already know that the volume of one hemisphere of a unit radius sphere is $2\pi/3$. Let's try to derive this result using the divergence theorem. Let the origin be at the center of a unit sphere and then compute the surface integral of the vector field $\vec{V} = \vec{r}$ over one hemisphere of the sphere (you may assume that the surface area of a unit sphere is $4\pi$). Explain how to use the divergence theorem to rewrite this surface integral as a volume integral. Calculate this volume integral and use this to find the volume of a unit radius hemisphere.
Q5 scratch/extra space (do not erase your scratch computations, they might earn partial credit):
Question 5
Let $\vec{V}$ be any smooth vector field. Show that

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0.$$
Q6 scratch/extra space (do not erase your scratch computations, they might earn partial credit):
Question 6
Consider a vector field $\vec{W}$ representing a linear flow where $\vec{W}$ points in the same direction and has the same length at every point in space. **Find** a vector field $\vec{V}$ such that $\vec{W}$ equals the curl of $\vec{V}$ (you may assume that your $x$-axis and $\vec{W}$ point in the same direction). **Draw** pictures representing $\vec{V}$ and $\vec{W}$. Also **draw** two closed curves $C_1$ and $C_2$ such that $\oint_{C_1} d\vec{r} \cdot \vec{V} = 0$ and $\oint_{C_2} d\vec{r} \cdot \vec{V} > 0$. Now **draw** two surfaces $\Sigma$ and $\Sigma'$ such that

$$\int_{\Sigma} dA \hat{N} \cdot \vec{W} = \int_{\Sigma'} dA \hat{N} \cdot \vec{W} = \oint_{C_2} d\vec{r} \cdot \vec{V}.$$  

Finally **draw** a surface $S$ for which $\int_{S} dA \hat{N} \cdot \vec{W} = 0$. **Indicate** on every surface in your pictures the direction of the unit normal vector $\hat{N}$, and on each closed curve the direction of integration.