Winter 2009: PhD Algebra Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1:
(a) Find a complex matrix \( M \) with

\[
M^2 = \begin{bmatrix}
1 & 3 & -3 \\
0 & 4 & 5 \\
0 & 0 & 9 \\
\end{bmatrix}
\]

(b) Note that \(-M\) is another solution. How many such matrices are there?

Problem 2:
Show that there are at least two nonisomorphic, non-Abelian groups of each of the orders 24 and 30.

Problem 3:
Show that if a finite group has exactly three conjugacy classes (noting that the identity forms one of the three) then the group has at most six elements. (Hint: Consider showing that in a finite group the number of elements in any conjugacy class divides the number of elements in the group.)

Problem 4: Let \( P \) be some set of prime numbers in the usual integers. Find a commutative ring \( R \) containing the integers such that the primes (irreducible elements) in \( R \) are precisely the elements of \( P \) up to multiplication by units. (Hint: What are all primes in the ring \( \mathbb{Z}[\frac{1}{2}] = \{ \frac{a}{2^n} | a \in \mathbb{Z}, 0 \leq r \in \mathbb{Z} \} \)?)

Problem 5: Let \( A \) be the group of rational numbers under addition, and let \( M \) be the group of positive rational numbers under multiplication. Determine all homomorphisms from \( A \) to \( M \).

Problem 6: An element of the ring of \( n \)-adic integers \( \mathbb{Z}_n \) is an infinite (to the left) string of base \( n \) digits written \( \ldots a_3a_2a_1a_0 \) with \( 0 \leq a_i \leq n - 1 \). Addition and multiplication are defined as usual for the integers written in base \( n \), except that one must carry indefinitely. (Note for example that in \( \mathbb{Z}_{10} \) the element \( \ldots 99999 \) is \(-1\).)

Prove that \( \mathbb{Z}_{10} \) is isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_5 \).