Fall 2006: MA Algebra Preliminary Exam

Instructions:

(1) Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

(2) Use separate sheets for the solution of each problem.

Problem 1. Let $G$ be a matrix group, and let $g \in G$ be an element with $\det(g) \neq 1$. Show that $g \notin G'$, the commutator group of $G$.

Problem 2. Let $A : V \to V$ be an operator on a finite-dimensional vector space $V$. Suppose $A$ has characteristic polynomial $x^2(x - 1)^4$ and minimal polynomial $x(x - 1)^2$. What is the dimension of $V$? What are the possible Jordan forms of $A$?

Problem 3. Show that $\mathbb{Z}$ is a principal ideal domain.

Problem 4. Let $G$ denote a finite abelian group. Let us consider the set $G^*$ of all homomorphisms of the group $G$ into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers.

(a) Check that $G^*$ can be considered as a group with respect to the operation of multiplication of homomorphisms.

(b) Prove that the group $G^*$ is isomorphic to the group $G$.

Problem 5. Let us assign to every nonsingular complex $2 \times 2$ matrix $A$ a transformation $\phi_A$ of the vector space $\text{Mat}_2$ of complex $2 \times 2$ matrices defined by the formula

$$\phi_A(X) = AXA^{-1}.$$ 

(a) Check that this formula specifies an action of the group $GL_2(\mathbb{C})$ of nonsingular complex matrices on $\text{Mat}_2$; moreover, it specifies a linear representation of this group.

(b) Prove that this representation is reducible.

Problem 6. Consider the dihedral group $D_9$ (the group of isometries of regular 9-gons).

(a) Write down a list of all elements of $D_9$.

(b) Prove that $D_9$ cannot be represented as a direct product of two non-trivial groups.

(c) Determine if $D_9$ is solvable.
Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a given continuous function. Let $T_k : C([0,1]) \to C([0,1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x,y)f(y) \, dy$.

(a) Show that $T_k$ is a bounded operator.

(b) Find an expression for $\|T_k\|$ in terms of $k$.

(c) What is $\|T_k\|$ if $k(x,y) = x^2y^3$?

Problem 2. Let $X$ be a metric space.

(a) Define $X$ is sequentially compact.

(b) Define $X$ is a complete metric space.

(c) Prove that a sequentially compact metric space $X$ is complete.

(d) Let $B = \{ x : \|x\|_2 \leq 1 \}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that $B$ is not sequentially compact.

Problem 3. Give an example of a Banach space $X$ and a sequence $(x_n)$ of elements in $X$ such that $\sum_{n=1}^{\infty} x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^{\infty} |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n \in \mathbb{Z}}$ be the Fourier coefficient sequence of $f$; here, $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that $f$ is continuous? (In other words, is there a continuous function that is equivalent to $f$ in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions $T$ of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^1)$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0,1])$ symmetric, essentially self-adjoint, self-adjoint? Justify your answers.

(a) $D_A = C^1[0,1]$ (the space of continuously differentiable complex-valued functions on $[0,1]$)

(b) $D_A = \{ f \in C^1[0,1] : f(0) = f(1) \}$

(c) $D_A = \{ f \in C^1[0,1] : f(0) = f(1) = 0 \}$