Fall 2006: MA Algebra Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let G be a matrix group, and let $g \in G$ be an element with $\det(g) \neq 1$. Show that $g \notin G'$, the commutator group of G.

Problem 2. Let $A : V \to V$ be an operator on a finite-dimensional vector space V. Suppose A has characteristic polynomial $x^2(x-1)^4$ and minimal polynomial $x(x-1)^2$. What is the dimension of V? What are the possible Jordan forms of A?

Problem 3. Show that $\mathbb{Z}$ is a principal ideal domain.

Problem 4. Let G denote a finite abelian group. Let us consider the set G^* of all homomorphisms of the group G into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers.

(a) Check that G^* can be considered as a group with respect to the operation of multiplication of homomorphisms.

(b) Prove that the group G^* is isomorphic to the group G.

Problem 5. Let us assign to every nonsingular complex 2×2 matrix A a transformation ϕ_A of the vector space Mat_2 of complex 2×2 matrices defined by the formula

$$\phi_A(X) = AXA^{-1}.$$

(a) Check that this formula specifies an action of the group $GL_2(\mathbb{C})$ of nonsingular complex matrices on Mat_2; moreover, it specifies a linear representation of this group.

(b) Prove that this representation is reducible.

Problem 6. Consider the dihedral group D_9 (the group of isometries of regular 9-gons).

(a) Write down a list of all elements of D_9.

(b) Prove that D_9 cannot be represented as a direct product of two non-trivial groups.

(c) Determine if D_9 is solvable.

Fall 2006: MA Analysis Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0,1]\times[0,1] \to \mathbb{R}$ be a given continuous function. Let $T_k : C([0,1]) \to C([0,1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x,y)f(y)\,dy$.

(a) Show that T_k is a bounded operator.

(b) Find an expression for $\|T_k\|$ in terms of k.

(c) What is $\|T_k\|$ if $k(x,y) = x^2y^3$?

Problem 2. Let X be a metric space.

(a) Define X is sequentially compact.

(b) Define X is a complete metric space.

(c) Prove that a sequentially compact metric space X is complete.

(d) Let $B = \{\, x : \|x\|_2 \le 1 \,\}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that B is not sequentially compact.

Problem 3. Give an example of a Banach space X and a sequence (x_n) of elements in X such that $\sum_{n=1}^\infty x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^\infty |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n\in\mathbb{Z}}$ be the Fourier coefficient sequence of f; here, $\mathbb{T} := \{\, z \in \mathbb{C} : |z| = 1 \,\}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that f is continuous? (In other words, is there a continuous function that is equivalent to f in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions T of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^1)$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0,1])$ *symmetric*, *essentially self-adjoint*, *self-adjoint*? Justify your answers.

(a) $D_A = C^1[0,1]$
(the space of continuously differentiable complex-valued functions on $[0,1]$)

(b) $D_A = \{\, f \in C^1[0,1] : f(0) = f(1) \,\}$

(c) $D_A = \{\, f \in C^1[0,1] : f(0) = f(1) = 0 \,\}$