Fall 2003 Mathematics Graduate Program MA Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. ANALYSIS

Problem 1. (a) For a function \( f : (a, b) \rightarrow \mathbb{R} \), \((a, b)\) an open interval, state briefly but precisely:

i. What is meant by the statement: \( f(x) \) is continuous at \( x_0 \in (a, b) \).

ii. What is meant by the statement: \( f(x) \) is continuous on \((a, b)\).

iii. What is meant by the statement: \( f(x) \) is uniformly continuous on \((a, b)\).

(b) Prove, directly from the definition, that the function \( f(x) = \frac{1}{x} \) is uniformly continuous on the interval \([1, \infty)\).

Problem 2. Let \( \{ U_n \}_{n=1}^\infty \) be a nested sequence of open sets in a topological space \( X \), so that \( U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1} \). Let \( x_n \in U_n \setminus U_{n-1} \). Set \( U = \cup_{n=1}^\infty U_n \). Prove that \{ \( x_n \) \} does not have a subsequence that converges to a point in \( U \).

Problem 3. Let \( T : (X, d) \rightarrow (X, d) \) be a contraction mapping from the metric space \( (X, d) \) to itself, so that for some \( r < 1 \), \( d(Tx, Ty) \leq rd(x, y) \) \( \forall x, y \in X \). Assume that \( x_0 \) is a fixed point of this mapping. Prove that \( d(x, x_0) \leq \frac{d(x, T(x))}{1-r} \).

Problem 4. Let \( y, y' \) be two elements of a Hilbert space \( H \). Prove that if \( \langle y, x \rangle = \langle y', x \rangle \) for every \( x \in H \) then \( y = y' \).

Problem 5. Let \( L \) and \( R \) be the left shift operator and the right shift operator of \( l^2(\mathbb{N}) \) respectively. So

\[
L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)
\]
\[
R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, x_4, \ldots).
\]

Find the point spectrum of \( L \) and \( R \).

Problem 6. Define the following three sequences of functions \([0, +\infty) \rightarrow \mathbb{R}\):

\[
(f_n)_{n=1}^\infty \text{ given by } f_n(x) = \begin{cases} 
\frac{n^{1/2}}{(x+1)^{1/2}} & \text{if } 0 \leq x \leq n \\
0 & \text{else}
\end{cases}
\]

\[
(g_n)_{n=1}^\infty \text{ given by } g_n(x) = \begin{cases} 
\sin(2\pi nx) & \text{if } n \leq x \leq n + 1 \\
0 & \text{else}
\end{cases}
\]

\[
(h_n)_{n=1}^\infty \text{ given by } h_n(x) = \sum_{k=1}^{n} \frac{k}{\sqrt{n}} \text{Ind}_{[(k-1/n), k]}(x).
\]

Consider these sequences with each of the topologies given below and determine whether or not they converge and, if they converge, determine their limits. Explain your assertions.

a. Pointwise on \([0, +\infty)\).

b. Uniformly on \([0, +\infty)\).

c. In the norm topology of \( L^2([0, +\infty)) \).

d. Strongly in \( L^{3/2}([0, +\infty)) \).

e. Weakly in \( L^{3/2}([0, +\infty)) \).
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2. Algebra and Linear Algebra

Problem 7. Let $G$ be a group and $p$ a prime. Prove or give a counter example:

(a) A group of order $p$ is commutative.
(b) A group of order $p^2$ is commutative.
(c) A group of order $p^3$ is commutative.

Problem 8. Let $F$ be a finite field. Show that the number of elements of $F$ is $p^r$ for some prime $p$ and positive integer $r$.

Problem 9. A vector space $V$ contains an $n$-element set with the following properties:

(i) It is not linearly independent, but contains an $(n-1)$-element linearly independent set;
(ii) It does not span $V$, but is contained in an $(n+1)$-element spanning set.

Prove that $\dim V = n$.

Problem 10. Let $B$ be a symmetric, non-degenerate, not positive definite bilinear form in an $n$-dimensional real vector space $V$. Prove that there exists a basis $v_1, \ldots, v_n$ in $V$ such that $B(v_i, v_i) < 0$ for all $i$.

Problem 11. Let $I \subset \mathbb{R}[x]$ be the ideal generated by the polynomial $x^2 + 2x + 3$. Prove that the quotient ring $\mathbb{R}[x]/I$ is isomorphic to the field $\mathbb{C}$ of complex numbers.

Problem 12. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find $\alpha$ so that $K = \mathbb{Q}(\alpha)$, and compute the irreducible polynomial of $\alpha$ over $\mathbb{Q}$.