Math 119A Final Winter 2008

1. (41 points) Consider the equation $\dot{x} = x - \frac{\alpha x}{1+x}$ with $\alpha > 0$ and $x > -1$.

   (a) Find the fixed points $x^*$ as functions of $\alpha$ and classify their stability.

   (b) Plot the bifurcation diagram using $\alpha$ as the control parameter and indicate the stability of fixed points on the diagram. At what values of $\alpha$ does a bifurcation occur? What type of bifurcation is it?

   (c) What type of bifurcation would occur if $\dot{x} = x - \frac{\alpha x}{1+x} + \beta$? And why?

2. (30 points)Consider the system: $\dot{x} = \sin y, \quad \dot{y} = \sin x$.

   (a) Show that the system is reversible.

   (b) Find all the fixed points and classify their stabilities by linear stability analysis if possible.

   (c) Sketch the phase portrait.

3. (60 points) Consider the initial value problem $\ddot{x} + x + \epsilon x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 0$.

   (a) Obtain the exact solution to the problem.

   (b) Find $x_0$ and $x_1$ in the straightforward perturbation method, i.e. $x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2)$.

   (c) Calculate the averaged equation and find the amplitude and frequency of any limit cycles for the original system. Solve the averaged equations explicitly to find $x(t, \epsilon)$ and show that it is consistent with the exact solution that you obtained in part a.

4. (45 points) Consider the nonlinear system of ODEs:

$$\begin{aligned} \dot{x} &= y - x((x^2 + y^2)^2 - \mu(x^2 + y^2) - 1) - 1) \\ \dot{y} &= -x - y((x^2 + y^2)^2 - \mu(x^2 + y^2) - 1) - 1) \end{aligned}$$

   (a) Rewrite the system in polar coordinates.

   (b) For $0 \le \mu < 1$, show that the circular region that lies within concentric circles with radius $r_{min} = 1/2$ and $r_{max} = 2$ is a trapping region. And use the Poincaré-Bendixson theorem to show that there exist a stable limit cycle.

   (c) Show that a Hopf Bifurcation occurs at fixed point $(0, 0)$ and $\mu = 1$. Is it Sup- or Sub-critical?

5. (24 points) Consider the vector field given in the polar coordinates by $\dot{r} = r - r^2, \dot{\theta} = 1$.

   (a) Compute the Poincaré map from $S$ to itself, where $S$ is the positive $x$-axis.

   (b) Show that the system has a unique periodic orbit and classify its stability.