All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $A$ and $B$ be two bounded, self-adjoint operators on a Hilbert space $\mathcal{H}$. Prove that
   \[ ||Af|| \cdot ||Bf|| \geq \frac{||[A,B]f,f||}{2}, \]
   where $[A,B] = AB - BA$ is the commutator of $A$ and $B$. In addition, prove that equality holds if and only if $Af = cBf$ for some $c \in \mathbb{R}$.

2. Let $f : \mathbb{T} \to \mathbb{C}$ be a $C^1$ function such that $\int_{-\pi}^{\pi} f(x)dx = 0$. Show that
   \[ \int_{-\pi}^{\pi} |f(x)|^2dx \leq \int_{-\pi}^{\pi} |f'(x)|^2dx. \]

3. Let $y = \{a_n\}_{n=1}^\infty$ be a sequence of real-valued scalars and assume that the series $\sum_{n=1}^\infty a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$. Show that $y \in \ell^2(\mathbb{N})$.

4. Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f}$, the Fourier transform of $f$, is supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let $\text{sinc}(x) = \frac{\sin x}{x}$. Prove that
   \[ f(x) = \sum_{n \in \mathbb{Z}} f(n)\text{sinc}(x - n), \]
   where the equality holds in the $L^2$-sense. (Here, the Fourier transform of a function $g$ is given by $\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i x \omega}dx$.)

5. Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function and fix $1 < p < \infty$. Given $f \in L^p([0, 1])$, define $Tf : [0, 1] \to \mathbb{R}$ by
   \[ Tf(x) = \int_0^1 K(x,y)f(y)dy. \]
   (a) Prove that $Tf$ is a continuous function.
   (b) Prove that the image under $T$ of the unit ball in $L^p([0, 1])$ is precompact in $C([0, 1])$.

6. Let $D$ denote the closed unit disk in $\mathbb{C}$, and consider the complex Hilbert space
   \[ \mathcal{H} \overset{\text{def}}{=} \left\{ f : D \to \mathbb{C} \left| f(z) = \sum_{k=0}^\infty a_k z^k \text{ and } ||f||^2_{\mathcal{H}} = \sum_{k=0}^\infty (1 + k^2)|a_k|^2 < \infty \right. \right\}. \]
   Prove that the linear functional $L : \mathcal{H} \to \mathbb{C}$ defined by $L(f) = f(1)$ is bounded, and find an element $g \in \mathcal{H}$ such that $L(f) = \langle g, f \rangle$. (In other words, so that $g$ represents $L$ as in the Riesz representation theorem.)